IT Licentiate theses
2006-009

Topology Optimization for Acoustic Wave Propagation Problems

EDDIE WADBRO

UPPSALA UNIVERSITY
Department of Information Technology

UPPSALA
UNIVERSITET

Topology Optimization for Acoustic Wave Propagation Problems

By

Eddie Wadbro

October 2006

Division of Scientific Computing
Department of Information Technology
Uppsala University
Uppsala
Sweden

Dissertation for the degree of Licentiate of Philosopy in Scientific Computing
at Uppsala University 2006

Topology Optimization for Acoustic Wave Propagation Problems

Eddie Wadbro

Eddie.Wadbro@it.uu.se

Division of Scientific Computing
Department of Information Technology
Uppsala University
Box 337
SE-751 05 Uppsala
Sweden

http://www.it.uu.se/

ISSN 1404-5117
Printed by the Department of Information Technology, Uppsala University, Sweden

Abstract

The aim of this study is to develop numerical techniques for the analysis and optimization of acoustic horns for time harmonic wave propagation. An acoustic horn may be viewed as an impedance transformer, designed to give an impedance matching between the feeding waveguide and the surrounding air. When modifying the shape of the horn, the quality of this impedance matching changes, as well as the angular distribution of the radiated wave in the far field (the directivity). The dimensions of the horns considered are in the order of the wavelength. In this wavelength region the wave physics is complicated, and it is hard to apply elementary physical reasoning to enhance the performance of the horn. Here, topology optimization is applied to improve the efficiency and to gain control over the directivity of the acoustic horn.

List of Papers

This thesis is a summary of the following papers. They will be referred to as Paper A, Paper B, and Paper C.

A E. Wadbro and M. Berggren. Topology optimization of an acoustic horn. *Computer Methods in Applied Mechanics and Engineering*, In Press, 2006. Corrected proof, available online, doi:10.1016/j.cma.2006.05.005.

B E. Wadbro and M. Berggren. Topology optimization of wave transducers. In M. P. Bendsøe, N. Olhoff, and O. Sigmund, editors, *IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials*, pages 301–310. Springer, 2006.

C E. Wadbro On the far-field properties of an acoustic horn. Technical Report 2006-042, Department of Information Technology, Uppsala University, 2006.

Acknowledgments

I am grateful to my advisor Martin Berggren for introducing me to this interesting subject and for the many fruitful discussions we have had. I also thank my colleagues and friends for creating a good atmosphere and supporting many interesting conversations. This work has been supported by the Swedish Research Council (Vetenskapsrådet).

Contents

1 Introduction

When designing an object it is usually desirable that the object is optimal in some sense. Optimality can refer to having the smallest energy loss, the most esthetically appealing design, or the highest expected rate of return. In this thesis only problems where the performance can be measured are examined. That is, it is assumed that the performance can be evaluated as a mathematical function known as the *objective function*.

A design is said to be optimal if the value of the objective function for the design is as low—or high, depending on the objective—as possible. Moreover, it is assumed that the physics governing the problem can be described mathematically by a so-called *state equation*. Usually, it is required that the final structure stays within a certain region of space, the *design region*, and it is also common that there are *constraints* limiting the choices of the designer. For example, when designing a load-bearing structure, it is desired that the structure should be as stiff as possible (objective), and it may also be required that the weight of the structure is at most a pre-specified quantity (constraint).

One way of finding an optimal design is through the use of computer simulations. In these simulations the geometry is parameterized, and the optimal design is sought in the parameter space. That is, the algorithm systematically evaluates the performance (the objective function) for different parameter combinations. Often the geometry is parameterized from a given *reference configuration* or *ground structure*. The computer program can only fully exploit the chosen parameterization, thus the resulting computed optimal designs may vary as the parameterization is changed. Three main branches in design optimization are *sizing optimization*, *boundary shape optimization*, and *topology optimization*.

Sizing optimization is used to find the optimal thicknesses (sizes) of the parts in a structure. In boundary shape optimization, geometry changes, given by boundary displacements of the reference configuration, are examined. Boundary shape optimization is appropriate in the final steps of the design process. The optimal design obtained using boundary shape optimization is topologically equivalent with the reference configuration. A more general approach is topology optimization, where the connectedness of the optimal design is determined as a part of the optimization.

Topology optimization for continuum structures is a relatively new research area and has developed rapidly during the last two decades.

The first problems considered were from structural mechanics, and the most popular problem is to minimize the compliance of a structure, that is, making it as stiff as possible for a given set of load conditions and a given maximal weight. During the last decade researchers have started to apply similar strategies also for problems in other disciplines, such as for fluids and wave propagation.

The focus of this thesis is to study and apply topology optimization to an acoustic horn operating as a wave transducer. The main characteristics of an acoustic horn are its *efficiency* and the *directivity* properties. The efficiency can be seen as a measure of the quality of the impedance matching between the feeding waveguide and the surrounding air, and the directivity describes the behavior of the wave propagation in the far field. One way of defining an optimal horn is to find the horn with the highest efficiency for a frequency or frequency range. The problem of optimizing an acoustic horn has been studied using boundary shape optimization by Bängtsson *et al.* [4, 9] and Udawalpola and Berggren [46].

In Paper A, topology optimization is applied to optimize the efficiency using the same physical dimensions of the horn as Bängtsson *et al.* A natural extension is to include also the far-field behavior of the radiated waves in the optimization. Paper B presents acoustic horns optimized both for efficiency and directivity, and Paper C establishes the mathematical theory behind the far-field properties and describes the numerical evaluation of the far-field pattern.

2 Topology Optimization

Topology optimization consists of finding the best distribution of material within a given region. The first examined topology optimization problem appears to be the design of truss structures transmitting specified loads to given points of support. Already in 1866 this problem was addressed by Culmann [17], who proposed a positioning strategy for the joints in the structure. In 1904 Michell [25] presented important principles for low volume truss structures, using infinitely many infinitely thin bars, that are optimal with respect to weight. With the introduction of computers, the development of numerical algorithms for finding the optimal design started. The ideas presented by Culmann and Michell were extended and generalized by among others Prager [32] and Rozvany and Prager [34]. In 1988, Bendsøe and Kikuchi [6] published their pioneering work on topology optimization for continuum structures. Since then, topology optimization has been subject to intense research, and is today used as part of the design process of advanced components, for instance in the car and aeronautical industries. For some classes of problems there are commercial software, from for example Altair Engineering and FE-Design. A short introduction to the subject, with emphasis on the methods and techniques used to optimize the acoustic horn, is given in this section. A review of topology optimization of continuum structures is given by Eschenauer and Olhoff [20]. For a exhaustive introduction to topology optimization and its applications, the interested reader is referred to the monograph [8] by Bendsøe and Sigmund.

2.1 The Material Distribution Method

The most common approach to topology optimization is to model the presence of material with a material indicator function α, such that $\alpha = 1$ where material is present and $\alpha = 0$ otherwise. The aim is to find a discrete-valued design that maximizes the performance, that is, minimizes (or maximizes) the value of the objective function. The state equation is here assumed to be given as a variational problem over a region Ω, that is

$$\begin{aligned} &\text{Find } u \in \mathcal{V} \text{ such that} \\ &\qquad a_\alpha(u, v) = \ell(v), \quad \forall\, v \in \mathcal{V}, \end{aligned} \tag{1}$$

where the *state variable* u is the physical property governed by the state equation, v is a test function, and $\mathcal{V}$ is an appropriate functional space henceforth denoted the *state space*. The objective function is denoted $J(\alpha)$ and the optimization problem is given by

$$\begin{aligned} \min_{\alpha \in \mathcal{U}} \quad & J(\alpha) \\ \text{subject to} \quad & a_\alpha(u, v) = \ell(v), \quad \forall\, v \in \mathcal{V}, \\ & \text{all constraints satisfied,} \end{aligned} \tag{2}$$

where $\mathcal{U}$ is the set of admissible designs. When $\mathcal{U}$ is the set of all functions α such that $\alpha \in \{0, 1\}$ almost everywhere in Ω, then the optimization problem above is in general a nonlinear integer programming problem. These problems are computationally expensive to solve; the seemingly simpler class of linear integer programming problems is NP-complete [28,

p. 358]. Another problem with the integer formulation is that, in many cases, the problems are ill-posed, in that there exist non-convergent minimizing sequences. The standard method used to clear this obstacle is to relax the problem and let α take values in the continuous range $\alpha \in [0, 1]$. For elastic structures, the relaxed material indicator function α can be interpreted as a relative density and is thus often denoted ρ.

2.2 Discretization

When the optimization problem is solved with a computer, the design domain is partitioned into small chunks (*elements*). The material function α is then approximated by a function $\alpha_h \in \mathcal{U}_h$. Usually the space $\mathcal{U}_h$ consists of all functions being constant on each element. The state equation is for example discretized using the finite element method (FEM). The discretized state equation is

$$\begin{aligned} &\text{Find } u_h \in \mathcal{V}_h \text{ such that} \\ &\quad a_{h,\alpha_h}(u_h, v) = \ell_h(v_h), \quad \forall\, v_h \in \mathcal{V}_h, \end{aligned} \tag{3}$$

where u_h, v_h, a_h, and ℓ_h are the discretized counterparts to u, v, a, and ℓ respectively, and $\mathcal{V}_h \subset \mathcal{V}$ is the discretized state space. The discretized optimization problem reads

$$\begin{aligned} \min_{\alpha_h \in \mathcal{U}_h} \quad & J_h(\alpha_h) \\ \text{subject to} \quad & a_{h,\alpha_h}(u_h, v) = \ell_h(v_h), \quad \forall\, v_h \in \mathcal{V}_h, \\ & \text{all constraints satisfied,} \end{aligned} \tag{4}$$

where J_h is a discrete version of the objective function. Problem (4) is often solved using a gradient based method. The method used to optimize the acoustic horns in Papers A and B is the method of moving asymptotes (MMA) [43]. For many topology optimization problems MMA is a good choice. By construction MMA handles box constraints; moreover, the convex approximations used by MMA are of a form that utilize the fact that the state and the design variable appear bilinearly in the state equation. For an efficient computation it is crucial to be able to correctly compute the gradient of the objective function efficiently.

2.3 Numerical and Mathematical Issues

This section contains a short presentation of different restriction methods used to treat the ill-posedness and the numerical instabilities that might occur. A systematic investigation of restriction methods together with illustrative numerical examples in elastic continua is presented by Borrvall [11] and an overview of numerical instabilities in topology optimization is given by Sigmund and Petersson [38]. In this section a small selection of the strategies to handle these problems is presented.

When the optimization problem is discretized with the finite element method, the zero lower bound on the design variables might cause that the state matrix becomes singular, and thus the state equation is not uniquely solvable. To avoid this problem, the bounds are often changed into $\alpha \in \{\varepsilon, 1\}$, where $\varepsilon > 0$, in the binary case, and to $\alpha \in [\varepsilon, 1]$ in the continuous case. The material indicator function α^*, corresponding to the computed optimal design for the relaxed problem, is frequently not a binary function. One way of dealing with this problem is to introduce a penalty so that the values 0 and 1 are promoted. Another problem is the ill-posedness of the binary problem, which causes that the solutions (of the discretized problem) depend on the size of the elements in the mesh, and do not converge as the mesh is refined. A third problem is that the numerically optimal design might have regions of checkerboard-like structures. This problem originates from at least one of: improper choice of finite elements in the discretization or nonuniqueness of solutions (in the continuum problem) [29]. These problems are normally treated by choosing the proper elements, filtering, adding a perimeter constraint, or by imposing gradient constraints.

2.3.1 Penalization

In topology optimization of linearly elastic structures, a common way of tackling the problem with the intermediate values is the SIMP (Solid Isotropic Material with Penalization) approach. This approach was suggested by Bendsøe [5] and is traditionally used for problems where the objective, for example, is to minimize the compliance of a structure with a constraint on the maximum weight of the structure. The idea is to introduce an artificial density function

$$f(\alpha) = \alpha^q, \tag{5}$$

where q is a constant, and to modify state equation (1) into

$$\begin{aligned} &\text{Find } u \in \mathcal{U} \text{ such that} \\ &\quad a_{f(\alpha)}(u, v) = \ell(v), \ \ \forall v \in \mathcal{V} \end{aligned}$$

while keeping the volume constraint, that is,

$$\int_{\Omega} \alpha \leq V, \tag{6}$$

where V is a pre-specified constant. By choosing $q \gg 1$ the elements with intermediate design values, that is $0 < \alpha < 1$, carry little stiffness compared to their weight.

Remark 1. Whenever there is no risk for confusion, as in equation (6), the symbol of measure will not be stated explicitly.

Another interpolation scheme RAMP (Rational Approximation of Material Properties), suggested by Rietz [33] and studied in more detail by Stolpe and Svanberg [40], is given by choosing the artificial density function f as

$$f(\alpha) = \frac{\alpha}{1 + q(1-\alpha)}, \tag{7}$$

where the parameter q describes the amount of penalization, $q = 0$ gives no penalization, and the intermediate values give less stiffness relative to weight as q is increased. Another material interpolation function suggested by Borrvall [11], is obtained by combining the interpolation functions (5) and (7) into the two-parameter function

$$f(\alpha) = \frac{\alpha^{q_1}}{1 + q_2(1-\alpha)}. \tag{8}$$

A number of interpolation functions, including SIMP, are analyzed and compared—RAMP had not seen the light of day when this comparison was made—by Bendsøe and Sigmund [7].

The material interpolation schemes are based on the fact that there is a constraint of the form (6). A more versatile approach to deal with the intermediate values is to explicitly add a penalty function J_p to the objective function in problem (2) and to examine the penalized problem

$$\begin{aligned} \min_{\alpha \in \mathcal{U}} \quad & J(\alpha) + J_p(\alpha) \\ \text{subject to} \quad & a_\alpha(u, v) = \ell(v), \ \ \forall v \in \mathcal{V}, \\ & \text{all constraints satisfied.} \end{aligned}$$

A similar approach is to include the penalty as a constraint and to examine the following problem:

$$\begin{aligned} \min_{\alpha \in \mathcal{U}} \quad & J(\alpha) \\ \text{subject to} \quad & a_\alpha(u,v) = \ell(v), \ \ \forall\, v \in \mathcal{V}, \\ & J_p(\alpha) \le C, \\ & \text{all constraints satisfied,} \end{aligned}$$

where C is a constant. A commonly used penalty function, suggested for topology optimization (in 1993) by Allaire and Kohn [2], and used by for instance Jog and Haber [24] and Borrvall and Petersson [12], is

$$J_p = \gamma \int_\Omega (\alpha - \varepsilon)(1 - \alpha),$$

where γ is a constant representing the amount of penalization that should be used.

The penalized problems might have many local minima, and the penalization also destroys the existence of solutions for the continuous problem. To increase the chance of obtaining good results—hopefully a global minimum—the penalization methods presented above are in practice often used together with a continuation approach.[1] That is, the optimization problem is first solved without penalty then the penalty parameter q or γ is increased and the problem is solved again, this time with the results of the unpenalized optimization as a starting guess. Then the penalty parameter is increased and the problem is once again solved, with the results from the preceeding optimization round as a starting guess. The process is then repeated, and usually this method works satisfactory. However, Stolpe and Svanberg [41] show that, for continuously increasing penalization, the trajectory of the global optimal solutions might be discontinuous, and give examples where the procedure fails to produce an ε–1 design no matter how large the penalization becomes.

2.3.2 Filter

The problems with checkerboards and mesh-dependence can be handled using a filtering technique. The technique is analogous to the filtering used for denoising in image processing. The use of filtering in topology optimization was suggested by Sigmund [36], who used a filter to stabilize

[1] This is standard procedure when using penalty methods in nonlinear optimization.

the numerical computations by modifying the design sensitivities. This was done by letting the sensitivity used in the computation be a weighted average of the sensitivities over the elements in a fixed neighborhood.

Another use of the filtering technique suggested, by Bruns and Tortorelli [14] is to let the filter act on the design instead of only the sensitivities. This technique optimizes over a function $\widetilde{\alpha}$ and lets the physical design α be defined through the convolution

$$\alpha(x) = \int_{\mathbb{R}^d} \Phi(x, y)\widetilde{\alpha}(y)\,\mathrm{d}y,$$

where d is the number of space dimensions. The kernel Φ often is chosen as a function of the kind

$$\Phi(x, y) = \sigma \max\left\{0, 1 - \left(\frac{|x-y|}{\tau}\right)^p\right\},$$

where $\tau > 0$ is the filter radius, p is a parameter describing the shape of the kernel, and σ a constant such that

$$1 = \int_{\mathbb{R}^d} \Phi(x, y)\,\mathrm{d}y.$$

The choice $p = 1$ gives the kernel a hat shape and is the one most used in practise. When the parameter p tends to infinity the kernel function tends to the characteristic function of the ball with radius τ (multiplied with σ), and the kernel function tends to the Dirac delta function when p (or τ) tends to zero. For the special choice $p = 1$ the minimum size of the structural parts in the filtered design is approximately twice the filter radius [11]. Thus, the filter radius can be used as a tool specifying the minimal size of the parts in the design. The filtered version of optimization problem (2) reads:

$$\begin{aligned} \min_{\widetilde{\alpha}\in\mathcal{U}} \quad & J(\alpha) \\ \text{subject to} \quad & \alpha(\cdot) = \int_{\mathbb{R}^d} \Phi(\cdot, y)\widetilde{\alpha}(y)\,\mathrm{d}y, \\ & a_\alpha(u, v) = \ell(v), \quad \forall\, v \in \mathcal{V}, \\ & \text{all constraints satisfied.} \end{aligned} \tag{9}$$

Bourdin [13] studies a filtered version, with a fixed kernel, of the minimum compliance topology optimization problem, proves existence of solutions, and shows that one obtains convergence of solutions to the finite element discretized version of the problem.

2.3.3 Gradient and Perimeter Control

In the binary setting, the material indicator function is equal to the characteristic function for a set $E_\alpha \subset \Omega$. Let $P(E_\alpha, \Omega)$ denote the perimeter of E_α in Ω. Ambrosio and Buttazzo [3] studied the problem of finding the minimal energy configuration for a mixture of two conducting materials and proved existence of solutions to a penalized problem of the form

$$\min_{\alpha \in \mathcal{U}} J(\alpha) + P(E_\alpha, \Omega).$$

A related approach is to instead limit the maximum allowed perimeter of the design by adding a constraint of the kind

$$P(E_\alpha, \Omega) \leq C, \tag{10}$$

where C is a constant, to problem (2). When α is allowed to take values in a continuous range the perimeter constraint above is replaced by a constraint on the variation of the design, that is

$$\sup_{\phi} \left\{ \int_\Omega \alpha \operatorname{div}\phi \;\; ; \;\; \phi \in \mathcal{C}_0^1(\Omega),\ \|\phi(x)\|_p \leq 1 \;\; \forall x \in \Omega \right\} \leq C. \tag{11}$$

For a binary function α and $p = 2$, expressions (10) and (11) are equivalent. The first numerical experiments on topology optimization with a constraint on the variation of the design were performed by Haber *et al.* [21] using $p = \infty$. The problem was later treated more theoretically by Petersson [30] who showed existence of solutions to a class of problems, including the penalized compliance optimization, using a constraint on the variation of the design.

A similar approach is to put a limit on the gradients of the design function, that is

$$\left| \frac{\partial \alpha}{\partial x_i} \right| \leq C, \text{ for } i = 1, 2, \ldots, d.$$

The above approach was used to find the optimal design of an elastic plate by Niordson [26] in 1983. A proof of existence of solutions was given by Petersson and Sigmund [31]. However, for the Cartesian grid discretized version of the optimization problem, this approach adds as many constraints as the number of internal edges in the discretization.

2.4 Alternative Methods

The *bubble method* treats the topology optimization problem by solving a sequence of boundary shape optimization problems. After each shape optimization step, the introduction of new holes (bubbles) is possible. The positioning of the new bubble is performed by finding its optimal position in the structure using so-called topological sensitivities. The method generates a number of possible topologies and a criteria for the maximum number of holes must follow external demands on the construction [19]. A related method is based on the level set method [27] for tracking moving boundaries and topology changes in the optimization. Sethian and Wiegmann [35] present elastic structures optimized using this approach, where the velocity of the boundary movement was depending on the stress on the boundaries. The holes could merge during the boundary movement and new holes were introduced by cutting away material along contour lines of the von Mises stress. Instead of relying on such heuristics, Allaire *et al.* [1] perform optimization using the level set method, with the the velocity of the boundary movement given by the shape derivative.

An interesting, but computationally expensive approach, is to keep the condition that the design α is binary throughout the discretization. Stolpe and Svanberg [42] show that a large class of non-linear 0–1 topology optimization problems can also be modeled as linear mixed 0–1 problems. The integer programming problems have also been attacked using standard algorithms in combinatorial optimization, such as branch-and-bound and local search [28]. Svanberg and Werme utilize a hierarchical method with a neighborhood search to deal with topology optimization of discretized load-carrying structures [44, 45]; and Stolpe [39] uses a branch-and-bound technique to find the global optimum of minimum weight truss problems.

2.5 Topology Optimization for Wave Problems

During the last decade much work has been focused on extending the ideas and methods used in topology optimization of structures to other fields. The first use of a material distribution method for wave propagation problems was to maximize band gaps, that is, ranges of frequencies in which waves cannot propagate. The first articles were by Cox and Dobson [15, 16] who applied topology optimization to maximize band

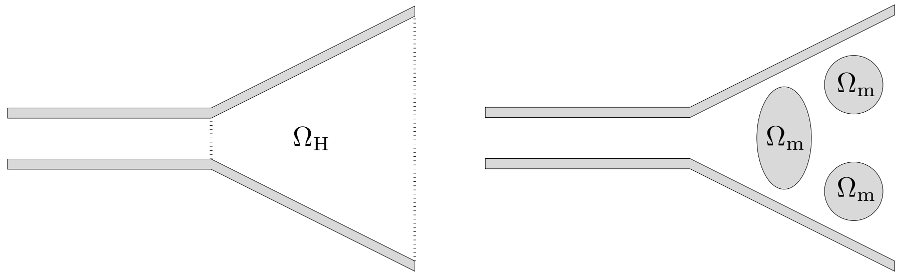


Figure 1: Material is placed in the region $\Omega_m \subset \Omega_H$ to improve the horn's radiation properties. The dotted lines in the left figure mark the left and right boundaries of the region Ω_H and are not part of the structure.

gaps in photonic crystals for E- and H-polarized light. The next type of wave propagation problems considered was the maximization of the transmission efficiency through different devices, like photonic waveguides [10, 22], or acoustical devices [23, 37]. A further extension for acoustic wave propagation, described in more detail in Papers B and C, is to optimize also over the far-field properties.

3 Problem Description

3.1 Problem Statement

The acoustic horn to be optimized is depicted in Figure 1. The geometry is assumed to be infinite in the direction normal to the plane. The wave transducer consists of a waveguide with a funnel-shaped termination (the horn). A wave propagating through the waveguide can be expressed as a superposition of modal components. For the studied problem it is possible to choose the width of the waveguide such that only the planar wave mode propagate.

Consider a single frequency planar wave moving from left to right in the waveguide. When this wave reaches the horn, parts of it will propagate out from the horn while other parts get reflected back into the waveguide. The transmission efficiency of the horn can be measured by comparing the amplitudes of the incoming right-going wave and the reflected left-going wave. The directivity describes the angular distribution of the radiated wave in the far field. When designing an acoustic horn, it is desired to have high transmission efficiency as well as control on the

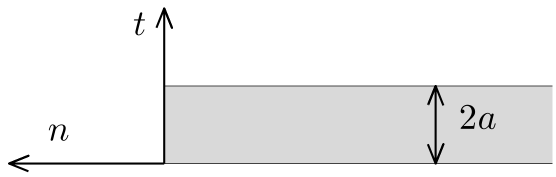


Figure 2: A truncated waveguide of width $2a$. The outward directed normal at the end of the waveguide is denoted n and t is orthogonal to n.

directivity of the horn.

The walls of the waveguide and the horn consist of a sound-hard solid material and that all other parts of space consist of air. The optimization consists of finding the region $\Omega_{\mathrm{m}} \subset \Omega_{\mathrm{H}}$ where solid material is to be placed to get the optimal performance of the horn. The presence of material in the region is modeled by the material indicator function α (the characteristic function of the region Ω_{m}).

3.2 The Acoustical Problem

Assume that the wave propagation is governed by the wave equation for the acoustical pressure P'

$$\frac{\partial^2 P'}{\partial t^2} = c^2 \Delta P',$$

where c is the speed of sound, and that all waves are outgoing in the far field. Seeking time harmonic solutions for a single frequency ω making use of the ansatz $P'(x,t) = \Re\{\mathrm{e}^{\mathrm{i}\omega t} p(x)\}$, where $\Re$ denotes the real part and i the imaginary unit, the above equation reduces to the following Helmholtz equation for the *complex amplitude function* p in the region filled with the fluid material,

$$c^2 \Delta p + \omega^2 p = 0. \tag{12}$$

Consider the truncated waveguide of width $2a$ illustrated in Figure 2. The waveguide is filled with air and the upper and bottom sides consist of sound-hard material. The wave propagation in the waveguide satisfies equation (12), and the boundary condition along the sound-hard walls is

$$\frac{\partial p}{\partial n} = 0.$$

The solution of this wave propagation problem for a wave of mode $m = 0, 1, \dots$ can be found by separation of variables and is

$$p(x) = \cos \frac{mx \cdot t\pi}{2a} \left(A_m e^{i\widehat{k}x\cdot n} + B_m e^{-i\widehat{k}x\cdot n} \right),$$

where A_m is a constant describing the complex amplitude of the incoming wave at the left end of the waveguide, B_m the amplitude of the outgoing wave, and $\widehat{k}$ is the reduced wavenumber in the waveguide given by

$$\widehat{k}^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{2a}\right)^2.$$

The wave of mode m propagates as long as $\widehat{k}^2 > 0$, or equivalently,

$$a > \frac{\pi cm}{2\omega} = \frac{cm}{4f}. \tag{13}$$

Hence by choosing the width of the waveguide sufficiently small only the planar mode ($m = 0$) propagates and the solution to equation (12) in the waveguide is

$$p(x) = A_0 e^{ikx\cdot n} + B_0 e^{-ikx\cdot n}, \tag{14}$$

where $k = \omega/c$ is the wavenumber. Differentiating expression (14) in the direction n results in

$$\frac{\partial p}{\partial n} = A_0 ik e^{ikx\cdot n} - B_0 ik e^{-ikx\cdot n}.$$

At the outer end of the truncated waveguide, $x \cdot n$ is constant and hence

$$ikp + \frac{\partial p}{\partial n} = 2A_0 ik e^{ikx\cdot n} = 2Aik, \tag{15}$$

where A is the amplitude of the incoming wave at the end of the waveguide. By imposing boundary condition (15), it is possible to set the amplitude of the incoming wave without affecting the amplitude of the outgoing wave.

The numerical solution of the acoustic wave propagation problem is performed on a bounded domain Ω shown in Figure 3 using the finite element method. When optimization is performed for the directivity of the horn the waveguide is truncated. This is done because the evaluation of the far-field properties requires the integration around a boundary that encloses all sound sources. The boundary Γ_{sym} is a symmetry boundary,

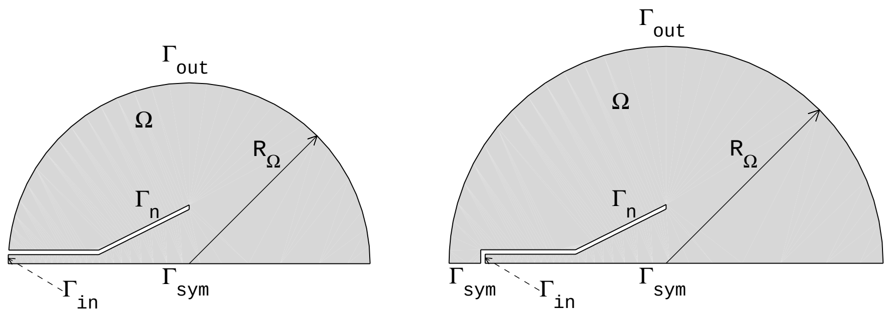


Figure 3: Computational domains. Left: domain used when optimizing only for efficiency and right: domain used when optimizing for both efficiency and directivity.

and Γ_{n} corresponds to the sound-hard boundaries of the waveguide and the horn. The inflow boundary Γ_{in} corresponds to a cross section of the waveguide and the outer boundary is denoted Γ_{out}.

Imposing a first-order outgoing wave boundary condition [18] on the outer boundary Γ_{out}, together with condition (15) on Γ_{in}, and a symmetry condition on Γ_{sym}, result in the following variational problem over Ω

Find $p \in H^1(\Omega)$ such that

$$c^2 \int_\Omega \alpha \nabla \bar{q} \cdot \nabla p - \omega^2 \int_\Omega \alpha \bar{q} p + \mathrm{i}\omega c \int_{\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}} \bar{q} p + \frac{c^2}{2R_\Omega} \int_{\Gamma_{\mathrm{out}}} \bar{q} p = 2\mathrm{i}\omega c A \int_{\Gamma_{\mathrm{in}}} \bar{q}, \quad \forall q \in H^1(\Omega).$$

In order for this problem to have a unique solution, as mentioned in Section 2.3, the lower bound for α is set to a small number ε.

3.3 The Optimization Problem

The two physical properties that are subject to optimization in Papers A and B are the efficiency and the directivity. To measure the efficiency, the mean complex amplitude at Γ_{in}, given by

$$\langle p \rangle_{\mathrm{in}} = \frac{1}{a} \int_{\Gamma_{\mathrm{in}}} p,$$

is observed. The reflection factor R is defined as the quotient between the amplitude B of the outgoing wave and the amplitude A of the ingoing

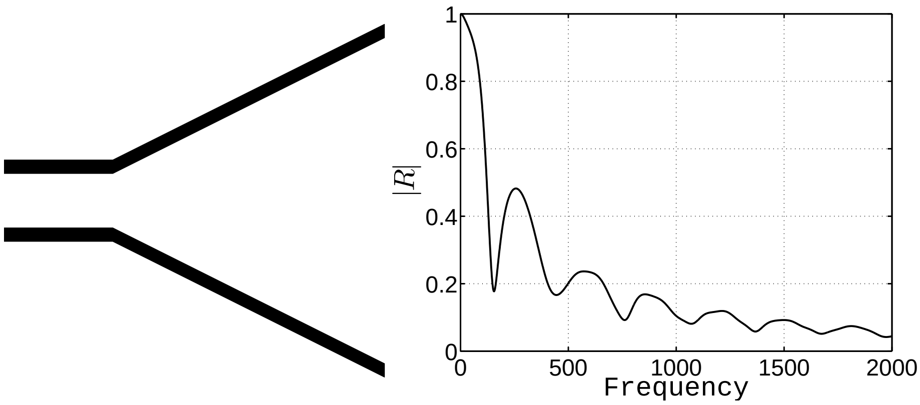


Figure 4: Left: the funnel shaped reference horn, right: the reflection spectrum of the reference horn.

wave, that is,

$$R = \frac{B}{A} = \frac{\langle p \rangle_{\text{in}} - A}{A}.$$

The reflection factor depends on the shape and size of the horn as well as the frequency of the incoming wave. Figure 4 depicts the funnel-shaped horn ($\alpha \equiv 1$ in Ω_{H}) and its reflection spectrum, that is, the absolute value of the reflection coefficient as a function of frequency. The dimensions of the funnel shaped reference horn are given in Table 1. From this table and equation (13), it follows that only the planar mode wave propagates ($c = 345$ m/s) for frequencies less than 1725 Hz.

Far from the horn, the complex amplitude function can be expanded as

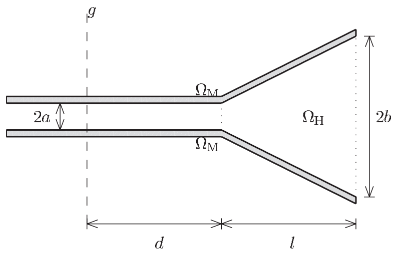
$$p(x) = \frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|^{(d-1)/2}} \left\{ p_\infty(\arg x) + O\left(\frac{1}{|x|}\right) \right\},$$

where $\arg x$ denote the argument(s) of x and d is the number of space dimensions considered. The function p_∞ is denoted the far-field pattern and characterizes the behavior of the wave propagation in the far field (Paper C).

The optimization problem is then formulated as the minimization of a function depending on the reflection factor for a single frequency or for a range of frequencies and the far-field pattern for these frequencies and some directions. For the numerical optimization, the problem is relaxed such that the indicator function α may take values in the continuous

Table 1: The dimensions of the horn—upper row the values used for the numerical experiments in Paper A, bottom row the values used in Paper B. The parameters a, b, l, and d are illustrated to the right, and R_Ω is the radius of the computational domain Ω (Figure 3).

a(m)	b(m)	l(m)	d(m)	R_Ω(m)
0.05	0.3	0.5	0.5	1
0.05	0.3	0.5	0.5	1.2



range $[\varepsilon, 1]$. The methods used to handle the numerical and mathematical issues when optimizing the acoustical horn are the explicit penalization method from Section 2.3.1 and the filtering of the design described in Section 2.3.2. The method of moving asymptotes is utilized to solve the resulting problem, and the required gradients are computed using the adjoint equations associated with the wave propagation problems.

4 Summary of Papers

4.1 Paper A

In Paper A the problem of maximizing the efficiency, or minimizing the absolute value of the reflection factor, of the horn is studied. The minimization problems are of the form

$$\min_{\alpha \in \mathcal{U}} \left\{ J_p + \sum_{\omega_j \in F} |\langle p^{(\omega_j)} \rangle_{\text{in}} - A|^2 \right\},$$

where J_p is a penalty function, F is the set of frequencies to optimize the horn for, and each pair ω_j, $p^{(\omega_j)}$ satisfies

$$\begin{aligned} &c^2 \int_\Omega \alpha \nabla \bar{q} \cdot \nabla p^{(\omega_j)} - \omega^2 \int_\Omega \alpha \bar{q} p^{(\omega_j)} \\ &+ \mathrm{i}\omega_j c \int_{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}} \bar{q} p^{(\omega_j)} + \frac{c^2}{2R_\Omega} \int_{\Gamma_{\text{out}}} \bar{q} p^{(\omega_j)} = 2\mathrm{i}\omega c A \int_{\Gamma_{\text{in}}} \bar{q}, \;\; \forall q \in H^1(\Omega). \end{aligned}$$

To compute the gradient efficiently the associated adjoint equations are solved—a detailed derivation of the gradient expressions are given in this paper. Paper A includes many numerical examples and presents horns optimized for different frequency ranges and using different order of basis functions. The numerical experiments also include an accuracy check where the reflection spectra for two of the optimized horns is computed using body fitted meshes for the final design.

The appendix presents an interpretation of the material indicator function α which applies to inviscid flow in general. The interpretation follows from considering the flow of a compressible, inviscid fluid between a planar surface T_0 and a surface $T_\alpha = \{x - \alpha(x) n_0 \,;\, x \in T_0\}$, where n_0 is the unit normal of T_0 and α is a positive real valued function representing the distance between the plane and the surface.

4.2 Paper B

Paper B can be seen as a direct continuation of Paper A. The problem examined in this paper also contains the far-field properties of the horn. The general form of the studied optimization problem is

$$\min_{\alpha \in \mathcal{U}} \left\{ \sum_{\omega_j} |\langle p^{(\omega_j)} \rangle_{\text{in}} - A|^2 \sum_{\omega_j, \theta_k} \beta_{j,k} |p_\infty^{(\omega_j)}(\theta_k)|^2 + J_p \right\},$$

where the constants $\beta_{j,k}$ can be adjusted to control the far-field behavior of the horn. The numerical experiments present horns, optimized with respect to the objective function above, together with their reflection spectra and polar plots of their far-field patterns.

4.3 Paper C

Paper C contains a theoretical treatment of the far-field behavior of the wave propagation and is intended to serve as a reference for work related to the directivity of the horn in two and three space dimensions. The paper presents a derivation for the time harmonic wave propagation and covers the numerical treatment of the far-field pattern in a few typical situations.

References

[1] G. Allaire, F. Jouve, and A.-M. Toader. Structural optimization using sensitivity analysis and a level-set method. *Journal of Computational Physics*, 194(1):363–393, 2004. doi:10.1016/j.jcp.2003.09.032.

[2] G. Allaire and R. V. Kohn. Topology optimization and optimal shape design using homogenization. In M. P. Bendsøe and C. A. Mota Soares, editors, *Topology design of structures*, pages 207–218. Kluwer Academic Publisher, 1993.

[3] L Ambrosio and G Buttazzo. An optimal design problem with perimeter penalization. *Calculus of Variations and Partial Differential Equation*, 1(1):55–69, 1993. doi:10.1007/BF02163264.

[4] E. Bängtsson, D. Noreland, and M. Berggren. Shape optimization of an acoustic horn. *Computer Methods in Applied Mechanics and Engineering*, 192:1533–1571, 2003. doi:10.1016/S0045-7825(02)00656-4.

[5] M. P. Bendsøe. Optimal shape design as a material distribution problem. *Structural Optimization*, 1:193–202, 1989. doi:10.1007/BF01650949.

[6] M. P. Bendsøe and N. Kikuchi. Generating optimal topologies in structural design using a homogenization method. *Computer Methods in Applied Mechanics and Engineering*, 71:197–224, 1988. doi:10.1016/0045-7825(88)90086-2.

[7] M. P. Bendsøe and O. Sigmund. Material interpolation schemes in topology optimization. *Archive of Applied Mechanics*, 69:635–654, 1999. doi:10.1007/s004190050248.

[8] M. P. Bendsøe and O. Sigmund. *Topology Optimization. Theory, Methods, and Applications*. Springer, 2003.

[9] M. Berggren, E. Bängtsson, and D. Noreland. Multifrequency shape optimization of an acoustic horn. In K. J. Bathe, editor, *Computational Fluid and Solid Mechanics 2003*, pages 2204–2207. Elsevier Science Ltd, 2003.

[10] P. Borel, A. Harpøth, L. Frandsen, M. Kristensen, J. S. Jensen P. Shi, and O. Sigmund. Topology optimization and fabrication of photonic crystal structures. *Optics Express*, 12(9):1996–2001, 2004.

[11] T. Borrvall. Topology optimization of elastic continua using restriction. *Archives of Computational Methods in Engineering*, 8(4):351–385, 2001.

[12] T. Borrvall and J. Petersson. Topology optimization using regularized intermediate density control. *Computer Methods in Applied Mechanics and Engineering*, 22:4911–4928, 2001. doi:10.1016/S0045-7825(00)00356-X.

[13] B. Bourdin. Filters in topology optimization. *International Journal for Numerical Methods in Engineering*, 50:2143–2158, 2001. doi:10.1002/nme.116.

[14] T. E. Bruns and D. A. Tortorelli. Topology optimization of non-linear elastic structures and compliant mechanisms. *Computer Methods in Applied Mechanics and Engineering*, 190:3443–3459, 2001. doi:10.1016/S0045-7825(00)00278-4.

[15] S. J. Cox and D. C. Dobson. Maximizing band gaps in two-dimensional photonic crystals. *SIAM Journal on Applied Mathematics*, 59(6):2108–2120, 1999. doi:10.1137/S0036139998338455.

[16] S. J. Cox and D. C. Dobson. Band structure optimization of two-dimensional photonic crystals in H-polarization. *Journal of Computational Physics*, 158(2):214–224, 2000. doi:10.1006/jcph.1999.6415.

[17] K. Culmann. *Die Graphische Statik*. Zürich, 1866.

[18] B. Engquist and A. Majda. Absorbing boundary conditions for numerical simulation of waves. *Mathematics of Computation*, 31(139):629–651, 1977. doi:10.2307/2005997.

[19] H. A. Eschenauer, V. V. Kobelev, and A. Schumacher. Bubble method for topology and shape optimization of structures. *Structural and Multidisciplinary Optimization*, 8(1):42–51, 1994. doi:10.1007/BF01742933.

[20] H. A. Eschenauer and N. Olhoff. Topology optimization of continuum structures: A review. *Applied Mechanics Reviews*, 54(4):331–390, 2001. doi:10.1115/1.1388075.

[21] R. B. Haber, C. S. Jog, and M. P. Bendsøe. A new approach to variable-topology shape design using a constraint on perimeter. *Structural Optimization*, 11(1–2):1–12, 1996. doi:10.1007/BF01279647.

[22] J. S. Jensen and O. Sigmund. Systematic design of photonic crystal structures using topology optimization: Low-loss waveguide bends. *Applied Physics Letter*, 84(12):2022–2024, 2004. doi:10.1063/1.1688450.

[23] J. S. Jensen and O. Sigmund. Systematic design of acoustic devices by topology optimization. In *Twelfth international congress on sound and vibration, ICVS12 2005, Lisbon*, 2005.

[24] C. S. Jog and R. B. Haber. Stability of finite element models for distributed-parameter optimization and topology design. *Computer Methods in Applied Mechanics and Engineering*, 130:203–226, 1996. doi:10.1016/0045-7825(95)00928-0.

[25] A. G. M. Michell. The limits of economy of material in frame-structures. *Philosophical Magazine Ser. 6*, 8(47):589–597, 1904.

[26] F. Niordson. Optimal design of elastic plates with a constraint on the slope of the thickness function. *International Journal of Solids and Structures*, 19(2):141–151, 1983. doi:10.1016/0020-7683(83)90005-7.

[27] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton–Jacobi formulations. *Journal of Computational Physics*, 79(1):12–49, 1988. doi:10.1016/0021-9991(88)90002-2.

[28] C. H. Papadimitriou and K. Steiglitz. *Combinatorial Optimization, Algorithms and Complexity.* Dover, 1998.

[29] J. Petersson. A finite element analysis of optimal variable thickness sheets. *SIAM Journal on Numerical Analysis*, 36(6):1759–1778, 1999. doi:10.1137/S0036142996313968.

[30] J. Petersson. Some convergence results in perimeter-controlled topology optimization. *Computer Methods in Applied Mechanics and Engineering*, 171(1–2):123–140, 1999. doi:10.1016/S0045-7825(98)00248-5.

[31] J. Petersson and O. Sigmund. Slope constrained topology optimization. *International Journal for Numerical Methods in Engineering*, 41(8):1417–1434, 1998. doi:10.1002/(SICI)1097-0207(19980430)41:8<1417::AID-NME344>3.0.CO;2-N.

[32] W. Prager. A note on discretized Michell structures. *Computer Methods in Applied Mechanics and Engineering*, 3(3):349–355, 1974. doi:10.1016/0045-7825(74)90019-X.

[33] A. Rietz. Sufficiency of a finite exponent in SIMP (power law) methods. *Structural and Multidisciplinary Optimization*, 21:159–163, 2001. doi:10.1007/s001580050180.

[34] G. I. N. Rozvany and W. Prager. Optimal design of partially discretized grillages. *Journal of the Mechanics and Physics of Solids*, 24(2–3):125–136, 1976. doi:10.1016/0022-5096(76)90022-3.

[35] J. A. Sethian and A. Wiegmann. Structural boundary design via level set and immersed interface methods. *Journal of Computational Physics*, 163(2):489–528, 2000. doi:10.1006/jcph.2000.6581.

[36] O. Sigmund. *Design of Material Structures Using Topology Optimization*. PhD thesis, Technical University of Denmark, 1994.

[37] O. Sigmund and J. S. Jensen. Design of acoustic devices by topology optimization. In In C. Cinquini, M. Rovati, P. Venini, and R. Nascimbene, editors, *Short papers of the 5th World Congress on Structural and Multidisciplinary Optimization WCSMO5. Lido de Jesolo, May, 2003, pp. 267–268*, 2003.

[38] O. Sigmund and J. Petersson. Numerical instabilities in topology optimization: a survey on procedures dealing with checkerboards, mesh-dependencies and local minima. *Structural Optimization*, 16(1):68–75, 1998. doi:10.1007/BF01214002.

[39] M. Stolpe. Global optimization of minimum weight truss topology problems with stress, displacement, and local buckling constraints using branch-and-bound. *International Journal for Numerical Methods in Engineering*, 61(8):1270–1309, 2004. doi:10.1002/nme.1112.

[40] M. Stolpe and K. Svanberg. An alternative interpolation scheme for minimum compliance topology optimization. *Structural and Multidisciplinary Optimization*, 22:116–124, 2001. doi:10.1007/s001580100129.

[41] M. Stolpe and K. Svanberg. On the trajectories of penalization methods for topology optimization. *Structural and Multidisciplinary Optimization*, 21(2):128–139, 2001. doi:10.1007/s001580050177.

[42] M. Stolpe and K. Svanberg. Modelling topology optimization problems as linear mixed 0–1 programs. *International Journal for Numerical Methods in Engineering*, 57(5):723–739, 2003. doi:10.1002/nme.700.

[43] K. Svanberg. The method of moving asymptotes—a new method for structural optimization. *International Journal for Numerical Methods in Engineering*, 24:359–373, 1987. doi:10.1002/nme.1620240207.

[44] K. Svanberg and M. Werme. A hierarchical neighbourhood search method for topology optimization. *Structural and Multidisciplinary Optimization*, 29(5):325–340, 2005. doi:10.1007/s00158-004-0493-x.

[45] K. Svanberg and M. Werme. Topology optimization by a neighbourhood search method based on efficient sensitivity calculations. *International Journal for Numerical Methods in Engineering*, To Appear, 2006. doi:10.1002/nme.1677.

[46] R. Udawalpola and M. Berggren. Optimization of an acoustic horn with respect to efficiency and directivity. *In review.*, 2006.

Recent licentiate theses from the Department of Information Technology

2006-008 Agnes Rensfelt: *Nonparametric Identification of Viscoelastic Materials*

2006-007 Stefan Engblom: *Numerical Methods for the Chemical Master Equation*

2006-006 Anna Eckerdal: *Novice Students' Learning of Object-Oriented Programming*

2006-005 Arvid Kauppi: *A Human-Computer Interaction Approach to Train Traffic Control*

2006-004 Mikael Erlandsson: *Usability in Transportation – Improving the Analysis of Cognitive Work Tasks*

2006-003 Therese Berg: *Regular Inference for Reactive Systems*

2006-002 Anders Hessel: *Model-Based Test Case Selection and Generation for Real-Time Systems*

2006-001 Linda Brus: *Recursive Black-box Identification of Nonlinear State-space ODE Models*

2005-011 Björn Holmberg: *Towards Markerless Analysis of Human Motion*

2005-010 Paul Sjöberg: *Numerical Solution of the Fokker-Planck Approximation of the Chemical Master Equation*

2005-009 Magnus Evestedt: *Parameter and State Estimation using Audio and Video Signals*

2005-008 Niklas Johansson: *Usable IT Systems for Mobile Work*

2005-007 Mei Hong: *On Two Methods for Identifying Dynamic Errors-in-Variables Systems*

2005-006 Erik Bängtsson: *Robust Preconditioned Iterative Solution Methods for Large-Scale Nonsymmetric Problems*

2005-005 Peter Nauclér: *Modeling and Control of Vibration in Mechanical Structures*

2005-004 Oskar Wibling: *Ad Hoc Routing Protocol Validation*

2005-003 Magnus Ågren: *High-Level Modelling and Local Search*

2005-002 Håkan Zeffer: *Hardware-Software Tradeoffs in Shared-Memory Implementations*

UPPSALA
UNIVERSITET

Department of Information Technology, Uppsala University, Sweden