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April 1999
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Rapport 99-001
ISSN 1404-0603
Institutionen för informationsteknologi

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February 25, 1999

Abstract

The singular values of a perturbed complex-valued matrix $A + \varepsilon B + O(\varepsilon^2)$ are shown to have singular values of the form $\sigma_i(\varepsilon) = \sigma_i + k_i\varepsilon + O(\varepsilon^2)$. Explicit expressions for the k_i coefficients are derived. The cases of zero singular values as well as multiple singular values are included in the analysis.

*Work supported in part by Swedish Research Council for Engineering Sciences, under contract 95-800.

1 Introduction and statement of the problem

Consider a complex-valued matrix $A(\varepsilon)$, of dimension $(m|n)$, $m \geq n$, that depends smoothly on a real-valued parameter ε. We are going to derive perturbation expressions for the singular values of the matrix in the case ε takes small values. The assumption $m \geq n$ it not restrictive, as we else can consider the matrix A^* which has the same singular values as A.

The problem has been treated in [8] for the case of real-valued matrices, using a different technique. The results presented here are quite similar to those of [8]. The case of a zero but distinct singular value has been analysed in [5].

More specifically, we introduce the following notations. The matrix is written as

$$\begin{aligned} A(\varepsilon) &= A + \varepsilon B(\varepsilon) && (1.1) \\ &= A + \varepsilon B + O(\varepsilon^2) && (1.2) \end{aligned}$$

Here, ε is assumed to be a small real-valued quantity, and $O(\varepsilon)/|\varepsilon|$ to be bounded, as ε approaches 0. Further, matrices without argument correspond to values for $\varepsilon = 0$:

$$A \triangleq A(0), \qquad B \triangleq B(0) = \frac{dA(\varepsilon)}{d\varepsilon}_{|\varepsilon=0} \tag{1.3}$$

The singular values of the matrix $A(\varepsilon)$ are denoted by $\sigma_i(\varepsilon)$, $i = 1, \ldots, n$. As singular values as well as eigenvalues depend continuously on the matrix elements, it holds that

$$\sigma_i(\varepsilon) = \sigma_i(0) + o(1) \triangleq \sigma_i + o(1) \tag{1.4}$$

where the remainder term $o(1)$ tends to zero as ε tends to zero. We will show that the singular values in fact satisfy, cf also [1], [2] and [10],

$$\begin{aligned} \sigma_i(\varepsilon) &= \sigma_i + k_i\varepsilon + O(\varepsilon^2) && (1.5) \\ k_i &= k_i(A, B) && (1.6) \end{aligned}$$

and give explicit expression for how the 'direction coefficients' k_i, $i = 1, \ldots, n$, can be computed from A and B.

The stated problem is of interest as such, not at least for quantifying sensitivity of the singular value decomposition. Many problems, for example in signals, systems and control, involve solving linear equations and/or performing eigendecompositions. Examples of such fundamental concepts are parameter estimation and pole-placement by state feedback. The sensitivity of the solutions is strongly coupled to the condition number of various matrices. If the stated problem is solved, we are given a tool for quantifying the smallest singular value of an almost rank-deficient matrix, which will be useful for analysing ill-conditioned problems.

For completeness, we give first some comments on the perturbed eigenvalue problem as well as the case of a distinct and strictly positive singular value.

2 Perturbation results for eigenvalues

It is well known that the singular values of $A(\varepsilon)$ are related to the eigenvalues of $A^*(\varepsilon)A(\varepsilon)$ as

$$\sigma_i^2(\varepsilon) = \sigma_i^2\left(A(\varepsilon)\right) = \lambda_i\left(A^*(\varepsilon)A(\varepsilon)\right) \tag{2.1}$$

It is therefore of some interest to first review results concerning perturbation of eigenvalues. This is a topic treated in several sources, including [4], [6]. The case of distinct eigenvalues is particular simple. Assume that λ_i is a distinct eigenvalue of A, with a right eigenvector u_i and a left eigenvector v_i^*:

$$Au_i = \lambda_i u_i, \qquad v_i^* A = v_i^* A \tag{2.2}$$

Then the corresponding eigenvalue of $A(\varepsilon)$ satisfies, see [4], [6],

$$\lambda_i(\varepsilon) = \lambda_i + \varepsilon \frac{v_i^* B u_i}{v_i^* u_i} + O(\varepsilon^2) \tag{2.3}$$

The case of non-distinct eigenvalues is more tricky. As an illustration consider the case

$$A(\varepsilon) = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \tag{2.4}$$

Clearly, $A = A(0)$ has both eigenvalues in 1. However, it is easily found that $A(\varepsilon)$ has eigenvalues

$$\lambda_i(\varepsilon) = 1 \pm \sqrt{\varepsilon} \tag{2.5}$$

Hence, we do not have a series expansion in ε in this case, as the eigenvalues in (2.5) are not differentiable for $\varepsilon = 0$. The reason for this behaviour is that the matrix A has a Jordan block of dimension higher than one (the dimension is two in this case).

For singular values there is still hope to have a more 'regular' behaviour for small values of ε, such as indicated in (1.5). The reason is that the singular values are related to a *symmetric* eigenvalue problem, see (2.1). Hermitian matrices (such as $A^*(\varepsilon)A(\varepsilon)$, for example) can always be diagonalized, and thus do not have any Jordan block of dimension higher than one. Perturbation results for multiple eigenvalues of symmetric matrices have been derived in [9], [10].

3 The case of distinct and positive singular values

We write the singular value decomposition of A in the following way:

$$A = \sum_{k=1}^{n} u_k \sigma_k v_k^* \tag{3.1}$$

where it holds that

$$u_j^* u_k = \delta_{j,k} \tag{3.2}$$
$$v_j^* v_k = \delta_{j,k} \tag{3.3}$$

and $\delta_{j,k}$ denotes the Kronecker delta.

The singular values, $\sigma_j(A)$ relate to the eigenvalues of $C \triangleq A^*A$ as, cf. (2.1),

$$\sigma_j^2(A) = \lambda_j(C), \qquad j = 1, \ldots, n \tag{3.4}$$

From (3.1) and (3.2) we can write an eigenvalue expansion of C as

$$\begin{aligned} C = A^*A &= \sum_j v_j \sigma_j u_j^* \sum_k u_k \sigma_k v_k^* \\ &= \sum_j v_j \sigma_j^2 v_j^* \end{aligned} \tag{3.5}$$

Assume that σ_j is distinct and strictly positive. By differentiating

$$\lambda_j(C(\varepsilon)) = \sigma_j^2(A(\varepsilon)) \tag{3.6}$$

cf. (3.4), we get

$$\dot{\lambda}_j(C) = 2\sigma_j \dot{\sigma}_j(A) \tag{3.7}$$

and hence, using (2.3),

$$\begin{aligned} \dot{\sigma}_j(A) &= \frac{1}{2\sigma_j} \frac{v_j^* \dot{C} v_j}{v_j^* v_j} \\ &= \frac{1}{2\sigma_j} v_j^* [\dot{A}^* A + A^* \dot{A}] v_j \\ &= \frac{1}{2\sigma_j} [v_j^* \dot{A}^* u_j \sigma_j + \sigma_j u_j^* \dot{A} v_j] \\ &= \Re[u_j^* \dot{A} v_j] = \Re[u_j^* B v_j] \end{aligned} \tag{3.8}$$

where $\Re[\cdot]$ denotes the real part of $[\cdot]$. Here we have used the fact, cf (3.1),

$$A v_j = u_j \sigma_j \tag{3.9}$$

The result (3.8) can also be found, for example, in [6], [7].

An alternative way of deriving the above result can be used, if we assume that both the singular values and the singular vectors are differentiable. First write the singular value decomposition for an arbitrary (small) value of ε as

$$A(\varepsilon) = \sum_{k=1}^{n} u_k(\varepsilon) \sigma_k(\varepsilon) v_k^*(\varepsilon) \tag{3.10}$$
$$u_j^*(\varepsilon) u_k(\varepsilon) = \delta_{j,k} \tag{3.11}$$
$$v_j^*(\varepsilon) v_k(\varepsilon) = \delta_{j,k} \tag{3.12}$$

By differentiating and then setting $\varepsilon = 0$, we get

$$\dot{A} = \sum_k \left(\dot{u}_k \sigma_k v_k^* + u_k \dot{\sigma}_k v_k^* + u_k \sigma_k \dot{v}_k^* \right) \tag{3.13}$$

and therefore,

$$u_j^* \dot{A} v_j = u_j^* \dot{u}_j \sigma_j + \dot{\sigma}_j + \sigma_j \dot{v}_j^* v_j \tag{3.14}$$

It further holds that

$$\dot{u}_j^* u_j + u_j^* \dot{u}_j = 0 \Rightarrow \Re(\dot{u}_j^* u_j) = 0 \tag{3.15}$$

Hence we get

$$\begin{aligned} \dot{\sigma}_j &= u_j^* \dot{A} v_j - u_j^* \dot{u}_j \sigma_j - \sigma_j \dot{v}_j^* v_j \\ &= \Re\left(u_j^* \dot{A} v_j\right) - \sigma_j \Re\left(u_j^* \dot{u}_j + \dot{v}_j^* v_j\right) \\ &= \Re\left(u_j^* \dot{A} v_j\right) \end{aligned} \tag{3.16}$$

This is the same result as (3.8).

4 The general case

4.1 Preliminaries

Let us now turn to the general case. As compared to the previous section there are two generalisations that are needed:

1. The case of multiple singular values.

2. The case of singular values identical to zero.

We will treat these two generalisations partly in different ways.

As a preparation we will rewrite the characterisation of the singular values. Assume for the general case that σ is a singular value (possibly equal to zero) of A with multiplicity r. We write the singular value decomposition of A as

$$A = \sigma U_1 V_1^* + U_2 \Sigma_2 V_2^* \tag{4.1}$$

where U_1 is $m|r$, V_1^* $r|n$, U_2 $m|(n-r)$, Σ_2 $(n-r)|(n-r)$ and V_2^* is $(n-r)|n$. Due to the above assumptions, it holds that the matrix $\Sigma_2 - \sigma I$ is nonsingular. Set

$$U = [U_1 \quad U_2] \qquad V = [V_1 \quad V_2] \tag{4.2}$$

Note that V is a unitary matrix. We further note that U_1 and V_1 in (4.1) are unique only up to an unknown unitary matrix. If U_1 and V_1 satisfy (4.1), so do $U_1 Q$ and $V_1 Q$, where Q is an arbitrary unitary matrix of order r.

The matrix $A(\varepsilon)$ will have r singular values close to σ. These singular values satisfy

$$\begin{aligned}
\sigma_i^2(\varepsilon) &= [\sigma_i(A(\varepsilon))]^2 \\
&= \lambda_i[A^*(\varepsilon)A(\varepsilon)] \\
&= \lambda_i[V^*A^*(\varepsilon)A(\varepsilon)V] \\
&= \lambda_i\left(\begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}\left(\sigma V_1 U_1^* + V_2\Sigma_2 U_2^*\varepsilon + B^*(\varepsilon)\right)\right. \\
&\quad \left.\times\left(\sigma U_1 V_1^* + U_2\Sigma_2 V_2^* + \varepsilon B(\varepsilon)\right)\begin{bmatrix} V_1 & V_2 \end{bmatrix}\right) \\
&= \lambda_i\left(\begin{bmatrix} \sigma U_1^* + \varepsilon V_1^* B^*(\varepsilon) \\ \Sigma_2 U_2^* + \varepsilon V_2^* B^*(\varepsilon) \end{bmatrix}\right. \\
&\quad \left.\times\begin{bmatrix} \sigma U_1 + \varepsilon B(\varepsilon)V_1 & U_2\Sigma_2 + \varepsilon B(\varepsilon)V_2 \end{bmatrix}\right) \\
&= \lambda_i\left(\begin{bmatrix} \sigma^2 I + \varepsilon\sigma(V_1^*B^*U_1 + U_1^*BV_1) + O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & \Sigma_2^2 + O(\varepsilon) \end{bmatrix}\right)
\end{aligned} \tag{4.3}$$

The matrix appearing in (4.3) is of dimension n, with the upper left block having dimension r. What is of interest is to investigate the r eigenvalues that are close to σ^2. As it turns out, such an analysis must be done in different ways depending on whether or not σ is strictly positive. In both cases we will make use of the following result.

Lemma 4.1. Let $A(\varepsilon)$ be Hermitian. Then the eigenvalues of the matrix $\sigma I + \varepsilon A(\varepsilon)$ satisfy

$$\lambda_i\left(\sigma I + \varepsilon A(\varepsilon)\right) = \sigma + \varepsilon\lambda_i(A) + O(\varepsilon^2) \tag{4.4}$$

Proof. Diagonalize $A = A(0)$ as $A = XDX^*$, $D = \mathrm{diag}(\lambda_i(A))$ and X unitary. Then $S \triangleq \sigma I + \varepsilon A(\varepsilon) = \sigma I + \varepsilon XDX^* + O(\varepsilon^2)$ has the same eigenvalues as

$$\tilde{S} = X^*SX = \sigma I + \varepsilon D + O(\varepsilon^2)$$

The Gerschgorin circles associated to $\tilde{S}$, which contain its eigenvalues, [6], are centered in $\sigma + \varepsilon\lambda_i(A)$ $i = 1, \ldots, r$, and have radii $O(\varepsilon^2)$. This observation completes the proof. ∎

4.2 Strictly positive singular values

For this case we will make use of the following lemma. It can also be found, in another form in some books, such as [6], [10].

Lemma 4.2. Consider the matrix

$$S = \begin{bmatrix} \sigma I_n + \varepsilon A(\varepsilon) & \varepsilon B(\varepsilon) \\ \varepsilon B^*(\varepsilon) & C + \varepsilon D(\varepsilon) \end{bmatrix} \tag{4.5}$$

where $A(\varepsilon)$, C and $D(\varepsilon)$ are Hermitian matrices, and $A(\varepsilon)$, $B(\varepsilon), D(\varepsilon)$ all are $O(1)$ as $\varepsilon \to 0$. Further, assume that $C - \sigma I$ is invertible. Then the n eigenvalues of S that for small values of ε are close to σ satisfy

$$\lambda_i = \sigma + \varepsilon\lambda_i(A(0)) + O(\varepsilon^2), \qquad i = 1, \dots, n \tag{4.6}$$

Proof. The characteristic equation of S can be written as, using results of the determinant for a partitioned matrix, see, for example, [3],

$$\begin{aligned} 0 &= \det \begin{bmatrix} sI - \sigma I - \varepsilon A(\varepsilon) & -\varepsilon B(\varepsilon) \\ -\varepsilon B^*(\varepsilon) & sI - C - \varepsilon D(\varepsilon) \end{bmatrix} \\ &= \det[sI - C + \varepsilon D(\varepsilon)] \\ &\quad \times \det \left[sI - \sigma I - \varepsilon A(\varepsilon) - \varepsilon B(\varepsilon) \left(sI - C - \varepsilon D(\varepsilon) \right)^{-1} \varepsilon B^*(\varepsilon) \right] \end{aligned} \tag{4.7}$$

For small values of ε there are precisely n solutions close to σ. Noting that $sI - C - \varepsilon D(\varepsilon)$ is invertible for small ε, we find that these eigenvalues must satisfy

$$\det \left[sI - \sigma I - \varepsilon A(\varepsilon) - \varepsilon B(\varepsilon) \left(sI - C - \varepsilon D(\varepsilon) \right)^{-1} \varepsilon B^*(\varepsilon) \right] = 0 \tag{4.8}$$

and hence

$$\det \left[sI - \sigma I - \varepsilon A(0) + O(\varepsilon^2) \right] = 0 \tag{4.9}$$

The result (4.6) then follows from Lemma 4.1. ∎

Next we apply Lemma 4.2 to (4.3), and get

$$\begin{aligned} \sigma_i^2(\varepsilon) &= \lambda_i \left[\sigma^2 I + \varepsilon\sigma(V_1^* B^* U_1 + U_1^* B V_1) \right] + O(\varepsilon^2) \\ &= \sigma^2 + \varepsilon\sigma\lambda_i \left(V_1^* B^* U_1 + U_1^* B V_1 \right) + O(\varepsilon^2) \end{aligned} \tag{4.10}$$

Thus,

$$\begin{aligned} \sigma_i(\varepsilon) &= \sigma \left[1 + \frac{\varepsilon}{\sigma}\lambda_i \left(V_1^* B^* U_1 + U_1^* B V_1 \right) + O(\varepsilon^2) \right]^{1/2} \\ &= \sigma \left[1 + \frac{\varepsilon}{2\sigma}\lambda_i \left(V_1^* B^* U_1 + U_1^* B V_1 \right) + O(\varepsilon^2) \right] \end{aligned} \tag{4.11}$$

Hence we have

Result 4.1. If $\sigma > 0$ has multiplicity r, it holds that

$$\sigma_i(\varepsilon) = \sigma + \frac{\varepsilon}{2}\lambda_i \left(V_1^* B^* U_1 + U_1^* B V_1 \right) + O(\varepsilon^2) \tag{4.12}$$
$$i = 1, \dots, r$$

We note that the matrix appearing in (4.12) is Hermitian, and hence λ_i is certainly real-valued. Further, it can be noted that the result (4.12) simplifies to our previous findings (3.8) in the case of a distinct and strictly positive singular value. Also, note that the result (4.12) does not depend on the unknown unitary matrix Q, cf. the discussion after (4.2).

4.3 Singular values being zero

We next consider the case when the singular value σ is identical to zero. If the result of the former subsection is applied, it will give

$$\sigma_i^2(\varepsilon) = O(\varepsilon^2) \tag{4.13}$$

which is not as detailed as we want. Further, we are not allowed to proceed as before, where we made some divisions by σ. In this case we have instead to take care of one additional term in the upper left block of the matrix in (4.3).

In the analysis we will later apply the following lemma.

Lemma 4.3. Consider the matrix

$$S = \begin{bmatrix} \varepsilon^2 A(\varepsilon) & \varepsilon B(\varepsilon) \\ \varepsilon B^*(\varepsilon) & C + \varepsilon D(\varepsilon) \end{bmatrix} \tag{4.14}$$

where $A(\varepsilon)$, C and $D(\varepsilon)$ are Hermitian matrices, $A(\varepsilon)$, $B(\varepsilon), D(\varepsilon)$ all are $O(1)$ as $\varepsilon \to 0$, and $A(\varepsilon)$ has dimension n. Further, assume that C is invertible. Then the n eigenvalues of S that for small values of ε are close to zero are given by

$$\lambda_i = \varepsilon^2 \lambda_i \left[A(0)) - B(0)C^{-1}B^*(0) \right] + O(\varepsilon^3), \qquad i = 1, \ldots, n \tag{4.15}$$

Proof. The characteristic equation of S can be written as, using again results for the determinant for a partitioned matrix,

$$\begin{aligned} 0 &= \det \begin{bmatrix} sI - \varepsilon^2 A(\varepsilon) & -\varepsilon B(\varepsilon) \\ -\varepsilon B^*(\varepsilon) & sI - C - \varepsilon D(\varepsilon) \end{bmatrix} \\ &= \det[sI - C - \varepsilon D(\varepsilon)] \\ &\quad \times \det \left[sI - \varepsilon^2 A(\varepsilon) - \varepsilon B(\varepsilon) \left(sI - C - \varepsilon D(\varepsilon) \right)^{-1} \varepsilon B^*(\varepsilon) \right] \end{aligned} \tag{4.16}$$

For small values of ε there are precisely n solutions close to $s = 0$. Noting that $sI - C - \varepsilon D(\varepsilon)$ then is invertible, we find that these eigenvalues must satisfy

$$\det \left[sI - \varepsilon^2 \left(A(0) - B(0)C^{-1}B^*(0) \right) + O(\varepsilon^3) \right] = 0 \tag{4.17}$$

Lemma 4.1 then gives (4.15). ■

We are now ready to reconsider (4.3) and to apply Lemma 4.3 in the analysis. In this case, when $\sigma = 0$, we have

$$A = U_2 \Sigma_2 V_2^*, \qquad \Sigma_2 \ \text{ nonsingular} \tag{4.18}$$

Repeating the calculations in (4.3), but taking (4.18) into account will lead to the following, where we now include some further terms, that turn out to be significant in this case.

$$\sigma_i^2(\varepsilon) \;=\; \lambda_i \left(\begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} [A^*(0) + \varepsilon B^*(\varepsilon)] \, [A(0) + \varepsilon B(\varepsilon)] \begin{bmatrix} V_1 & V_2 \end{bmatrix} \right)$$

$$
\begin{aligned}
&= \lambda_i \left(\begin{bmatrix} \varepsilon V_1^* B^*(\varepsilon) \\ \Sigma_2 U_2^* + \varepsilon V_2^* B^*(\varepsilon) \end{bmatrix} \begin{bmatrix} \varepsilon B(\varepsilon) V_1 & U_2\Sigma_2 + \varepsilon B(\varepsilon) V_2 \end{bmatrix} \right) \\
&= \lambda_i \left(\begin{bmatrix} \varepsilon^2 V_1^* B^*(0) B(0) V_1 + O(\varepsilon^3) & \varepsilon V_1^* B^*(0) U_2 \Sigma_2 + O(\varepsilon^2) \\ \varepsilon \Sigma_2 U_2^* B(0) V_1 + O(\varepsilon^2) & \Sigma_2^2 + O(\varepsilon) \end{bmatrix} \right)
\end{aligned} \tag{4.19}
$$

Applying Lemma 4.3 gives

$$
\begin{aligned}
\sigma_i^2(\varepsilon) &= \varepsilon^2 \lambda_i \left(V_1^* B^*(0) B(0) V_1 - V_1^* B^*(0) U_2 \Sigma_2 (\Sigma_2^2)^{-1} \Sigma_2 U_2^* B(0) V_1 \right) + O(\varepsilon^3) \\
&= \varepsilon^2 \lambda_i \left(V_1^* B^* [I - U_2 U_2^*] B V_1 \right) + O(\varepsilon^3) \\
&= \varepsilon^2 \lambda_i \left(V_1^* B^* U_1 U_1^* B V_1 \right) + O(\varepsilon^3)
\end{aligned} \tag{4.20}
$$

We can summarize the findings as

Result 4.2. In case $\sigma = 0$ has multiplicity r, it holds that

$$
\sigma_i(\varepsilon) = \varepsilon \sigma_i(U_1^* B V_1) + O(\varepsilon^2) \tag{4.21}
$$

$$
= \varepsilon \sqrt{\lambda_i \left(V_1^* B^* U_1 U_1^* B V_1 \right)} + O(\varepsilon^2), \qquad \varepsilon \to 0, \quad i = 1, \ldots, r \tag{4.22}
$$

5 Illustration

Consider the (real-valued) case

$$
A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \tag{5.1}
$$

The matrix A has rank one and one strictly positive singular value, for which

$$
\sigma = \sqrt{6}, \qquad u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
$$

Further, $u^* B v = -\sqrt{6}/3$, and hence the largest singular value of $A(\varepsilon)$ satisfies

$$
\sigma(\varepsilon) = \sqrt{6} - \frac{\sqrt{6}}{3} \varepsilon + O(\varepsilon^2) \tag{5.2}
$$

The matrix A has also two zero singular values. To find the corresponding small singular values of $A(\varepsilon)$ we apply (4.13). We choose the left and right singular subspaces as

$$
U_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}
$$

Some straightforward calculations then give

$$V_1^* B^* U_1 U_1^* B V_1 = \frac{2}{3} \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 2 \end{pmatrix}$$

which has eigenvalues in $1 \pm \sqrt{3}/3$. Therefore the two small perturbed singular values of $A(\varepsilon)$ are

$$\sigma(\varepsilon) = \varepsilon\sqrt{1 \pm \frac{\sqrt{3}}{3}} + O(\varepsilon^2) \tag{5.3}$$

As a graphical illustration, we show in Figure 1 how well the asymptotic expressions describe the singular values of $A + \varepsilon B$. To emphasize the perturbation effects we display the fractions

$$\frac{\sigma_i(A + \varepsilon B) - \sigma_i(A)}{\varepsilon}$$

as functions of ε. Due to the derived results, the fractions should approach the constants $-\sqrt{6}/3 \approx -0.82$, $\sqrt{1 + \sqrt{3}/3} \approx 1.26$ and $\sqrt{1 - \sqrt{3}/3} \approx 0.65$, as $\varepsilon \to 0$.

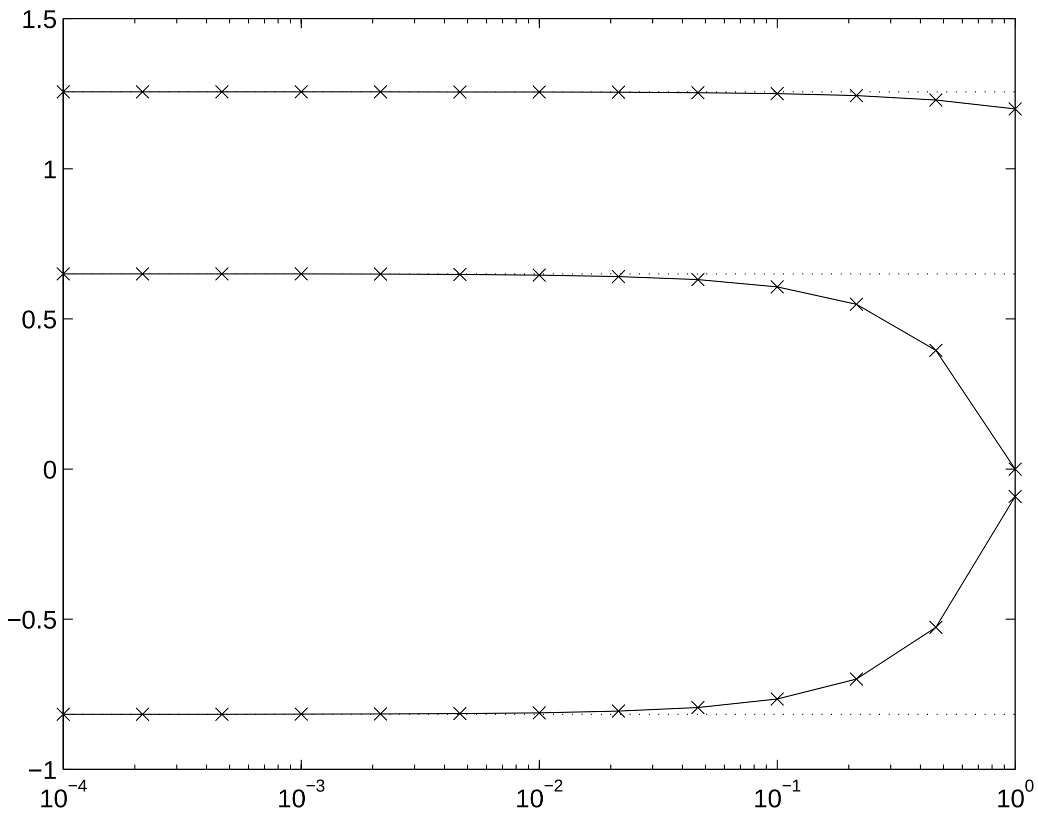

Figure 1: The ratios $[\sigma_i(A + \varepsilon B) - \sigma_i(A)]/\varepsilon$ versus ε.

6 Acknowledgement

The author is grateful to Professors Ji-gung Sun and Per-Åke Wedin for some useful hints, and for making the reference [10] available.

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