A consistent stabilized formulation for a nonsymmetric saddle-point problem

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Abstract

In this report we study the stability of a nonsymmetric saddle-point problem. It is discretized with equal order finite elements and stabilized with a consistent regularization. In this way we achieve a stable finite element discretization of optimal order approximation properties.

1 Introduction

We consider the problem of finding a vector function $\mathbf{u}(\mathbf{x})$ and a scalar function $p(\mathbf{x})$, $\mathbf{x} \in \Omega \subset \mathbb{R}^d, d = 2, 3$, which are the solutions of the following coupled system of partial differential equations

$$\begin{cases} -2\mu\Delta\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u}) - \nabla(\mathbf{u}\cdot\mathbf{b}) + \mathbf{c}(\nabla\cdot\mathbf{u}) & -\mu\nabla p & = \mathbf{f}(\mathbf{x}) \\ \mu\nabla\cdot\mathbf{u} & -\frac{\mu^2}{\lambda}p & = 0, \end{cases} \tag{1}$$

where μ and λ are scalar problem parameters which are piecewise constants in Ω. For simplicity of the presentation we will assume they are constant over the whole domain, which does not sacrifice the generality of the results. Details on the origin of the problem (1) can be found in [2], where it describes the isostatic response of the Earths lithosphere to glaciation and deglaciation. In that particular case

$$\mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \mu\frac{2\nu}{1-2\nu}$$

are the Lamé coefficients, where E is the Youngs modulus and ν is the Poisson number.

We aim at solving (1) numerically, using the finite element method. To this end, we consider the corresponding variational formulation:

$$\text{Seek } \mathbf{u}\in\mathbf{V} \text{ and } p\in P \text{ such that} \\ \begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \mathbf{f}(\mathbf{v}), & \forall\mathbf{v}\in\mathbf{V}, \\ b(\mathbf{u},q) - c(p,q) = 0, & \forall q\in P, \end{cases} \tag{2}$$

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where the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $m(\cdot,\cdot)$ are as follows

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_\Omega \Big[2\mu \textstyle\sum_{k=1}^d \nabla u_k \cdot \nabla v_k - \mu(\nabla \times \mathbf{u}) \cdot (\nabla \times \mathbf{v}) \\ &\quad -\nabla(\mathbf{u}\cdot\mathbf{b})\cdot\mathbf{v} + (\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})\Big] \\ b(\mathbf{v},p) &= \int_\Omega \mu(\nabla\cdot\mathbf{v})p = -\int_\Omega \mu\mathbf{v}\cdot\nabla p = -b(p,\mathbf{v}) \\ c(p,q) &= \int_\Omega \frac{\mu^2}{\lambda} pq \end{aligned}$$

Above, μ and λ are scalars (problem dependent parameters) which are assumed to be piecewise constant on Ω. Further it is assumed that $\mathbf{b}(\mathbf{x}) \in \mathbb{R}^d$ and $\mathbf{c}(\mathbf{x}) \in \mathbb{R}^d$ are vector fields such that there exist positive constants α_1, α_2 and β, independent on $\mathbf{u}$ and $\mathbf{v}$, for which there holds

$$|b_i(\mathbf{x})| \leq \alpha_1 \quad i = 1, \cdots, d \tag{3}$$

$$|\nabla\cdot\mathbf{b}| \leq \alpha_2 \tag{4}$$

$$|\mathbf{c}| \leq \beta \tag{5}$$

In [2] it was shown that the bilinear form $a(\mathbf{u},\mathbf{v})$ is bounded but not always coercive, and inequalities (6) and (7) hold true.

$$|a(\mathbf{u},\mathbf{v})| \quad \leq \quad C^{(0)}\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \qquad \text{(boundedness)} \tag{6}$$

Further, for all $\mathbf{u} \in \mathbf{V}$,

$$a(\mathbf{u},\mathbf{u}) \geq C^{(1)}\|\mathbf{u}\|_1^2 - C^{(2)}\|\mathbf{u}\|_0^2, \qquad \text{(Gårding-type relation)} \tag{7}$$

where $C^{(1)} > 0$ and $C^{(2)} > 0$ do not depend on $\mathbf{u}$, $C^{(0)} = 2K_1 + d(\alpha_1 + \alpha_2 + \beta)$, and K_1 is the coercivity constant for the bilinear form $\widetilde{a}(\mathbf{u},\mathbf{v}) = \int\limits_\Omega \left[2\mu \sum\limits_{k=1}^{d} (\underline{\nabla} u_k)\cdot(\underline{\nabla} v_k) - \mu(\underline{\nabla}\times\mathbf{u})\cdot(\underline{\nabla}\times\mathbf{v})\right] d\Omega.$

Note 1: The bilinear form $\widetilde{a}(\mathbf{u},\mathbf{v})$ corresponds to the classical linear elasticity problem and is coercive.

Note 2: For the limit value $\lambda = \infty$, which corresponds to an incompressible material, the bilinear form $c(p,g)$ vanishes.

2 Preliminaries and general theoretical results

In this Section, we highlight some results on the solvability and stability of saddle-point problems of the form of Equation (2) for Poisson number $\nu \in [0, 0.5]$.

For compressible materials (with Poisson number $\nu \in [0, 0.5)$), the solvability of Equation (2) is assured by the coercivity of the bilinear forms $a(\mathbf{u},\mathbf{v})$ and $c(p,q)$ together with boundedness of $a(\mathbf{u},\mathbf{v})$, $b(\mathbf{u},p)$, and $c(p,q)$, i.e.

$$a(\mathbf{u},\mathbf{v}) \quad \leq \quad \overline{a}\|\mathbf{u}\|_\mathbf{V}\|\mathbf{v}\|_\mathbf{V} \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{V} \tag{8}$$

$$b(\mathbf{v},p) \quad \leq \quad \overline{b}\|\mathbf{v}\|_\mathbf{V}\|p\|_P \quad \forall \mathbf{u} \in \mathbf{V},\, p \in P \tag{9}$$

$$c(p,q) \quad \leq \quad \overline{c}\|p\|_P\|q\|_P \quad \forall p,q \in P. \tag{10}$$

$$\tag{11}$$

As long as $a(\mathbf{u},\mathbf{u}) \geq \underline{a}\|\mathbf{u}\|_{\mathbf{V}}$, $\underline{a} > 0$, and $c(p,p) \geq \underline{c}\|p\|_P$, $\underline{c} > 0$, a solution $[\mathbf{u}, p] \in \mathbf{V} \times P$ exists and is unique.

When the material is incompressible, i.e $\lambda = \infty$, $c(p,q)$ vanishes and Equation (2) takes the form

$$\text{Seek } \mathbf{u} \in \mathbf{V} \text{ and } p \in P \text{ such that}$$
$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \mathbf{f}(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) = 0, & \forall q \in P. \end{cases} \tag{12}$$

As is shown, for example in [3], Equation (12) is solvable if $a(\mathbf{u},\mathbf{u})$ is coercive on the null-space of $b(\mathbf{u},q)$, and if

$$b(\mathbf{u},q) = 0 \quad \Rightarrow q = 0 \quad \forall \mathbf{u} \in \mathbf{V}.$$

Furthermore, Equation (12) is stable if the following inf-sup conditions are fulfilled,

$$\inf_{\mathbf{u}\in\mathbf{V}} \sup_{\mathbf{v}\in\mathbf{V}} \frac{a(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathbf{V}}\|\mathbf{v}\|_{\mathbf{V}}} \geq \underline{a}' > 0, \tag{13}$$

and

$$\inf_{q\in P} \sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{u},q)}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_P} \geq \underline{b} > 0. \tag{14}$$

Note that when $a(\mathbf{u},\mathbf{v})$ is coercive, Equation (13) is automatically satisfied.

To choose finite dimensional function spaces $\mathbf{V}^h$ and P^h satisfying the Ladyzhenskaya-Babuška-Brezzi (or LBB) condition in Equation (14) is not a trivial task, and one way to circumvent the discrete LBB-condition is to stabilize Equation (12). This can, for example, be done if $c(p,q)$ in Equation (2) is replaced by a regularized bilinear form $\widetilde{c}(p,q) = c(p,q) + d(p,q)$, where $d(q,q) > \underline{d}\|q\|_P$ regardless of the problem parameters.

As the problem after stabilization is on the form of Equation (2), with $c(p,q) \neq 0$ for $\nu \in [0, 0.5]$, the solvability of the stabilized system is assured by the coercivity of $a(\mathbf{u},\mathbf{v})$ and $\widetilde{c}(p,q)$.

In Section 3 - 5 optimal error bounds for a consistently stabilized equal order finite element discretization are derived without imposing the discrete analog of the inf-sup condition in Equation (14) on $b(\mathbf{u},p)$.

For these error bounds we assume that the solution $[\mathbf{u},p]$ of Equation (1) is bounded by the given data, i.e.

$$|\mathbf{u}|_2 + |p|_1 \leq \|\mathbf{f}\|_0. \tag{15}$$

We also assume that the condition (15) holds also for the solution of the dual problem to Equation (1), which in variational form reads,

$$\text{Find } \eta \in \mathbf{V} \text{ and } \xi \in P \text{ such that}$$
$$\begin{cases} a(\alpha,\eta) + b(\alpha,\xi) = \tilde{\mathbf{f}}(\alpha) & \forall \alpha \in \mathbf{V} \\ b(\eta,\zeta) - c(\xi,\zeta) = \tilde{g}(\zeta) & \forall \zeta \in P, \end{cases} \tag{16}$$

that is

$$|\eta|_2 + |\xi|_1 \leq \|\tilde{g}\|_0 + \|\tilde{\mathbf{f}}\|_0.$$

Note that since $a(\mathbf{v}, \mathbf{u}) \neq a(\mathbf{u}, \mathbf{v})$, the left hand sides of Equation (2) and Equation (16) are different.

The remaining issue is to give conditions under which the bilinear form $a(\mathbf{u}, \mathbf{v})$ guarantees solvability.

On the lack of coercivity of the bilinear form $a(\cdot, \cdot)$

Next we discuss how to handle the lack of coercivity of $a(\mathbf{u}, \mathbf{v})$. The coercivity estimate we need to show is of the form

$$a(\mathbf{u}, \mathbf{u}) \geq C\|\mathbf{u}\|_1^2, \quad \forall \mathbf{u} \in V. \tag{17}$$

The first (trivial) case is to make the assumption that $C^{(1)} - C^{(2)} \geq C^{(\delta)} > 0$. Then (17) is fulfilled with $C = C^{(\delta)}$.

Second, as noted in [2], $a(\mathbf{u}, \mathbf{v})$ is coercive for some particular vector fields $\mathbf{b}$ and $\mathbf{c}$, such as for example when $\nabla \cdot \mathbf{u}$ is zero (for purely incompressible materials) and in addition $\mathbf{b} = \mathbf{e}_d$, which entails $\nabla \cdot \mathbf{b} = 0$.

In the general case, where $\mathbf{b}$ and $\mathbf{c}$ are arbitrary, it is possible to show that under the assumption that the variational problem (2) possesses a unique nontrivial solution, $a(\mathbf{u}, \mathbf{v})$ must be coercive. Therefore, in the sequel we assume that condition (17) holds.

3 Stabilization

To stabilize Equation (2), we (scalar) multiply the first row of Equation (1) with ∇q, $q \in H_0^1(\Omega)$, and obtain the identity

$$\begin{aligned} -2\mu\Delta\mathbf{u} \cdot \nabla q - \mu\nabla \times (\nabla \times \mathbf{u}) \cdot \nabla q - \nabla(\mathbf{b} \cdot \mathbf{u}) \cdot \nabla q \\ + (\nabla \cdot \mathbf{u})(\mathbf{c} \cdot \nabla q) - \mu\nabla q \cdot \nabla p = \mathbf{f} \cdot \nabla q. \end{aligned} \tag{18}$$

Integration of Equation (18) over Ω yields

$$\begin{aligned} -\int_\Omega \mu\nabla q \cdot \nabla p = \int_\Omega \mathbf{f} \cdot \nabla q + \sum_{\tau_k} \int_{\tau_k} 2\mu\Delta\mathbf{u} \cdot \nabla q \\ + \int_\Omega \nabla(\mathbf{b} \cdot \mathbf{u}) \cdot \nabla q - \int_\Omega (\nabla \cdot \mathbf{u})(\mathbf{c} \cdot \nabla q), \end{aligned} \tag{19}$$

or

$$-d(p, q) = (\mathbf{f}, \nabla q) + e(q, \mathbf{u}; \mathbf{b}, \mathbf{c}). \tag{20}$$

The bilinear forms in Equation (20) are

$$\begin{aligned} d(p, q) &= \int_\Omega \mu\nabla p \cdot \nabla q \\ e(q, \mathbf{u}; \mathbf{b}, \mathbf{c}) &= \sum_{\tau_k} \int_{\tau_k} 2\mu\Delta\mathbf{u} \cdot \nabla q + \int_\Omega \mu\nabla \times (\nabla \times \mathbf{u}) \cdot \nabla q \\ &\quad + \int_\Omega \nabla(\mathbf{b} \cdot \mathbf{u}) \cdot \nabla q - \int_\Omega (\nabla \cdot \mathbf{u})(\mathbf{c} \cdot \nabla q), \end{aligned} \tag{21}$$

where τ_k denotes the kth finite element in the discretization Ω_h of Ω, i.e. $\Omega_h\ bigcup_k \tau_k$.

Multiply Equation (20) with a stabilization parameter σ, and add the result to the second row of Equation (2), that is

Find $\mathbf{u} \in \mathbf{V}$ and $p \in P$ such that

$$\left\{ \begin{array}{rcll} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) & = & \mathbf{f}(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) - c(p, q) - \sigma d(p, q) & = & \sigma(\mathbf{f}, \nabla q) & \\ & + & e(q, \mathbf{u}; \mathbf{b}, \mathbf{c}) & \forall q \in P. \end{array} \right. \tag{22}$$

The problem in Equation (22) is consistent with that in Equation (2) for any value of σ.

4 Error bounds in H^1-norm

In this section, we derive an upper bound on the discretization error for displacements and pressure in the Sobolev space $H^1(\Omega)$, i.e. $\|\mathbf{u}_h - \mathbf{u}\|_1$ and $\|p_h - p\|_1$. The derivation closely follows [1].

Consider the following discrete counterpart of Equation (22) where $\mathbf{u}_h$ and p_h are piecewise linear approximations of $\mathbf{u}$ and p, respectively, $\mathbf{u}$ and p,

Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\left\{ \begin{array}{rcll} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & \langle \mathbf{g}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{V}_h \\ b(\mathbf{u}_h, q_h) - c(p_h, q_h) - \sigma d(p_h, q_h) & = & & \\ \sigma(\mathbf{f}, \nabla q_h) + \sigma e(q_h, \mathbf{u}_h^*; \mathbf{b}, \mathbf{c}) & & & \forall q_h \in P^h, \end{array} \right. \tag{23}$$

where $\mathbf{u}_h^*$ is an approximation of $\mathbf{u}$.

Remark 4.1 *For constant vector fields* $\mathbf{b}$ *and* $\mathbf{c}$*, the choice* $\mathbf{u}_h^* = \mathbf{u}_h$ *induces* $e(q_h, \mathbf{u}_h^*; \mathbf{b}, \mathbf{c}) = 0$.

We substitute $\mathbf{v}$ and q in Equation (22) by $\mathbf{v}_h$ and q_h, and subtract Equation (23). The result reads

$$\left\{ \begin{array}{ll} a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) = \mathbf{0} & \forall \mathbf{v}_h \in \mathbf{V}^h \\ b(\mathbf{u} - \mathbf{u}_h, q_h) - c(p - p_h, q_h) - \sigma d(p - p_h, q_h) = & \\ \qquad \sigma e(q_h, \mathbf{u} - \mathbf{u}_h^*; \mathbf{b}, \mathbf{c}) & \forall q_h \in P^h, \end{array} \right. \tag{24}$$

Now, add $a(\mathbf{u}_I, \mathbf{v}_h)$ and $b(\mathbf{v}_h, p_I)$ to the first row of Equation (24), and add $b(\mathbf{u}_I, q_h)$, $-c(p - p_I, q_h)$ and $-\sigma d(p_I, q_h)$ to the second row. The interpolants $\mathbf{u}_I = \Pi_{\mathbf{V}_h}^{\mathbf{V}} \mathbf{u}$ and $p_I = \Pi_{P_h}^{P} p$ are defined by the interpolation operators

$$\Pi_{\mathbf{V}_h}^{\mathbf{V}} : \mathbf{V} \to \mathbf{V}_h \qquad \text{and} \qquad \Pi_{P_h}^{P} : P \to P_h.$$

After some manipulations, the resulting equation system is

$$
\begin{cases}
a(\mathbf{u}_I - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_I - p_h) \\
\qquad\qquad = a(\mathbf{u}_I - \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, p_I - p) \\
b(\mathbf{u}_I - \mathbf{u}_h, q_h) - c(p_I - p_h, q_h) - \sigma d(p_I - p_h, q_h) \\
\qquad\qquad = -\sigma e(q_h, \mathbf{u} - \mathbf{u}_h^*; \mathbf{b}, \mathbf{c}) + b(\mathbf{u}_I - \mathbf{u}, q_h) \\
\qquad\qquad\qquad - c(p_I - p, q_h) - \sigma d(p_I - p, q_h)
\end{cases} \tag{25}
$$

$\mathbf{V}_h$ is a linear space and $\mathbf{u}_I - \mathbf{u}_h \in \mathbf{V}_h$. Equation (25) holds $\forall \mathbf{v}_h \in \mathbf{V}_h$, and we can take $\mathbf{v}_h = \mathbf{u}_I - \mathbf{u}_h$. Similarly, we let $q_h = p_I - p_h \in P$, and subtract of the second row of Equation (25) from the first row,

$$
\begin{aligned}
&a(\mathbf{u}_I - \mathbf{u}_h, \mathbf{u}_I - \mathbf{u}_h) + c(p_I - p_h, p_I - p_h) \\
&+ \sigma d(p_I - p_h, p_I - p_h) = a(\mathbf{u}_I - \mathbf{u}_h, \mathbf{u}_I - \mathbf{u}) \\
&+ c(p_I - p, p_I - p_h) + \sigma d(p_I - p, p_I - p_h) - b(\mathbf{u}_I - \mathbf{u}, p_I - p_h) \\
&+ b(\mathbf{u}_I - \mathbf{u}_h, p_I - p) + \sigma e(p_I - p_h, \mathbf{u} - \mathbf{u}_h^*; \mathbf{b}, \mathbf{c}).
\end{aligned} \tag{26}
$$

Next, using the triangular inequality and Youngs inequality, we bound the different bilinear forms in Equation (26) as follows

$$
\begin{array}{lcl}
a(\mathbf{v}, \mathbf{w}) & \leq & 2K_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 + d\alpha \|\mathbf{v}\|_1 \|\mathbf{w}\|_0 + \beta \|\mathbf{v}\|_0 \|\mathbf{w}\|_1 \\
 & = & 2K_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 + d\alpha \|\mathbf{v}\|_1 \|\mathbf{w}\|_0 + \beta \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \\
c(q, r) & \leq & \frac{\mu^2}{\lambda} \|q\|_0 \|r\|_0 \\
d(q, r) & \leq & \mu \|q\|_1 \|r\|_1 \\
b(\mathbf{v}, q) & \leq & |b(\mathbf{v}, q)| \leq \mu \|\mathbf{v}\|_1 \|q\|_0 \\
-b(\mathbf{v}, q) & \leq & |b(\mathbf{v}, q)| \leq) \mu \mathbf{v}\|_1 \|q\|_0,
\end{array} \tag{27}
$$

where $\alpha = \alpha_1 + \alpha_2$.

For the last bilinear term, $e(q, \mathbf{v}; \mathbf{b}, \mathbf{c})$, there holds

$$
\begin{aligned}
e(q, \mathbf{v}; \mathbf{b}, \mathbf{c}) \leq{}& 2\mu \|q\|_1 \sum_k \|\Delta \mathbf{v}\|_{L^2(\tau_k)} \\
&+ d(\alpha_1 + \alpha_2) \|\mathbf{v}\|_1 \|q\|_1 + \beta d \|\mathbf{v}\|_1 \|q\|_1 \\
={}& 2\mu \|q\|_1 \sum_k \|\Delta \mathbf{v}\|_{L^2(\tau_k)} \\
&+ d\alpha \|\mathbf{v}\|_1 \|q\|_1 + \beta d \|\mathbf{v}\|_1 \|q\|_1.
\end{aligned} \tag{28}
$$

The left-hand side of Equation (26) is bounded from below by the coercivity estimate

- $C\|\mathbf{u}\|_1^2 \leq a(\mathbf{u}, \mathbf{u})$, which follows from Equation (17).
- $C^{(4)})\|p\|_0^2 \leq c(p, p)$, which follows by assumption
- $C^{(3)})\|p\|_1^2 \leq d(p, p)$, which follows from the definition of the bilinear form $d(p, q)$.

These lower bounds are combined with the combined with the upper bounds in Equation (27) and (28). When Youngs inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2$ are applied on the latter, Equation (26) reads

$$
\begin{aligned}
&C\|\mathbf{u}_I-\mathbf{u}_h\|_1^2+C^{(3)}\sigma\|p_I-p_h\|_1^2+C^{(4)}\|p_I-p_h\|_0^2 \le \\
&\epsilon_1 2\mu K_1\|\mathbf{u}_I-\mathbf{u}_h\|_1^2+\frac{\mu K_1}{2\epsilon_1}\|\mathbf{u}_I-\mathbf{u}\|_1^2)+\epsilon_2 d\alpha\|\mathbf{u}_I-\mathbf{u}_h\|_1^2 \\
&+\frac{d\alpha}{4\epsilon_2}\|\mathbf{u}_I-\mathbf{u}\|_0^2+\epsilon_3 d\beta\|\mathbf{u}_I-\mathbf{u}_h\|_1^2+\frac{d\beta}{4\epsilon_3}\|\mathbf{u}_I-\mathbf{u}\|_1^2 \\
&+\epsilon_4\frac{\mu^2}{\lambda}\|p_I-p\|_0^2+\frac{1}{4\epsilon_4}\frac{\mu^2}{\lambda}\|p_I-p_h\|_0^2+\epsilon_5\sigma\mu\|p_I-p\|_1^2 \\
&+\frac{\sigma\mu}{4\epsilon_5}\|p_I-p_h\|_1^2+\epsilon_6\mu\|\mathbf{u}_I-\mathbf{u}\|_1^2+\frac{\mu}{4\epsilon_6}\|p_I-p_h\|_0^2 \\
&+\epsilon_7\mu\|\mathbf{u}_I-\mathbf{u}_h\|_1^2+\frac{\mu}{4\epsilon_7}\|p_I-p\|_0^2+\epsilon_8 2\sigma\mu\|p_I-p_h\|_1^2 \\
&+\frac{\sigma\mu}{2\epsilon_8}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}^2+\epsilon_9\sigma\alpha d\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\
&+\frac{\sigma\alpha d}{4\epsilon_9}\|p_I-p_h\|_1^2+\epsilon_{10}\sigma\beta d\|\mathbf{u}-\mathbf{u}_h^*\|_1^2+\frac{\sigma\beta d}{4\epsilon_{10}}\|p_I-p_h\|_1^2.
\end{aligned}
\tag{29}
$$

Summation of the terms on the right-hand side gives

$$
\begin{aligned}
&C\|\mathbf{u}_I-\mathbf{u}_h\|_1^2+C^{(3)}\sigma\|p_I-p_h\|_1^2+C^{(4)}\|p_I-p_h\|_0^2 \le \\
&\le \{2K_1\mu\epsilon_1+\epsilon_2 d\alpha+\epsilon_3 d\beta+\epsilon_7\mu\}\|\mathbf{u}_I-\mathbf{u}_h\|_1^2 \\
&+\left\{\frac{\sigma\mu}{4\epsilon_5}+\epsilon_8 2\sigma\mu+\frac{\sigma d\alpha}{4\epsilon_9}+\frac{\sigma d\beta}{4\epsilon_{10}}\right\}\|p_I-p_h\|_1^2 \\
&+\left\{\frac{1}{4\epsilon_4}\frac{\mu^2}{\lambda}+\frac{\mu}{4\epsilon_6}\right\}\|p_I-p_h\|_0^2 \\
&+\left\{\frac{K_1\mu}{2\epsilon_1}+\frac{d\beta}{4\epsilon_3}+\epsilon_6\mu\right\}\|\mathbf{u}_I-\mathbf{u}\|_1^2+\frac{d\alpha}{4\epsilon_2}\|\mathbf{u}_I-\mathbf{u}\|_0^2 \\
&+\epsilon_5\sigma\mu\|p_I-p\|_1^2+\left\{\epsilon_4\frac{\mu^2}{\lambda}+\frac{\mu}{4\epsilon_7}\right\}\|p_I-p\|_0^2 \\
&+\{\epsilon_9\sigma d\alpha+\epsilon_{10}\sigma d\beta\}\|\mathbf{u}-\mathbf{u}_h^*\|_1^2+\frac{\sigma\mu}{4\epsilon_8}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}^2
\end{aligned}
\tag{30}
$$

When the ϵ-parameters are chosen as

$$
\begin{aligned}
&\epsilon_1=\frac{C}{16K_1\mu}\quad \epsilon_2=\frac{C}{8d\alpha}\quad \epsilon_3=\frac{C}{8d\beta}\quad \epsilon_4=\frac{1}{C^{(4)}}\frac{\mu^2}{\lambda} \\
&\epsilon_5=\frac{2\mu}{C^{(3)}}\quad \epsilon_6=\frac{\mu}{C^{(4)}}\quad \epsilon_7=\frac{C}{8\mu}\quad \epsilon_8=\frac{C^{(3)}}{16\mu} \\
&\epsilon_9=\frac{2d\alpha}{C^{(3)}}\quad \epsilon_{10}=\frac{2d\beta}{C^{(3)}},
\end{aligned}
\tag{31}
$$

the inequality in Equation (29) after suitable rearrangements becomes,

$$
\begin{aligned}
&\frac{1}{2}C^{(1)}\|\mathbf{u}_I-\mathbf{u}_h\|_1^2 - C^{(2)}\|\mathbf{u}_I-\mathbf{u}_h\|_0^2 \\
&\qquad\qquad + \frac{1}{2}C^{(3)}\sigma\|p_I-p_h\|_1^2 + \frac{1}{2}C^{(4)}\|p_I-p_h\|_0^2 \leq \\
&+ \left\{\frac{8K_1^2\mu^2}{C} + \frac{(d\beta)^2}{C} + \frac{\mu^2}{C}\right\}\|\mathbf{u}_I-\mathbf{u}\|_1^2 + \frac{(d\alpha)^2}{C}\|\mathbf{u}_I-\mathbf{u}\|_0^2 \\
&+ \sigma\frac{2\mu^2}{C^{(3)}}\|p_I-p\|_1^2 + \left\{\frac{1}{C^{(4)}}\left(\frac{\mu^2}{\lambda}\right)^2 + \frac{2}{C}\mu^2\right\}\|p_I-p\|_0^2 \\
&+ \sigma\left\{\frac{2}{C^{(3)}}(d\alpha)^2 + \frac{2}{C^{(3)}}(d\beta)^2\right\}\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\
&+ \sigma\frac{4\mu^2}{C^{(3)}}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}^2 \\
&= C^{(5)}\|\mathbf{u}_I-\mathbf{u}\|_1^2 + C^{(6)}\|\mathbf{u}_I-\mathbf{u}\|_0^2 + \sigma C^{(6)}\|p_I-p\|_1^2 \\
&+ C^{(7)}\|p_I-p\|_0^2 + \sigma C^{(8)}\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 + \sigma C^{(9)}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}^2
\end{aligned}
\tag{32}
$$

Applying standard interpolation theory we see that

$$
\begin{aligned}
&\frac{1}{2}C\|\mathbf{u}_I-\mathbf{u}_h\|_1^2 \\
&\qquad\qquad + \frac{1}{2}C^{(3)}\sigma\|p_I-p_h\|_1^2 + \frac{1}{2}C^{(4)}\|p_I-p_h\|_0^2 \leq \\
&\leq (h^2C^{(5)} + h^4C^{(6)})|\mathbf{u}|_2^2 + (h^2\sigma C^{(6)} + h^4C^{(7)})|p|_2^2 \\
&+ \sigma C^{(8)}\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 + \sigma C^{(9)}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L_2(\tau_k)}^2 \\
&(\leq \mathcal{O}(h^2) + \mathcal{O}(h^4) + \sigma\mathcal{O}(1) + \mathcal{O}(h^2))\|\mathbf{f}\|_0^2 + \sigma C^{(8)}\|\mathbf{u}-\mathbf{u}_h^*\|_1^2 \\
&+ \sigma C^{(9)}\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L_2(\tau_k)}^2,
\end{aligned}
\tag{33}
$$

where we utilize Equation (15) $|\mathbf{u}|_2^2 + |p|_1^2 \leq \|\mathbf{f}\|^2$, and the inverse estimate $|p|_2 \leq ch^{-1}(1+h)|p|_1$.

Thus, if we choose $\sigma = \mathcal{O}(h^2)$ and assume that

$$
\|\mathbf{u}-\mathbf{u}_h^*\|_1 = \mathcal{O}(1) \quad \text{and} \quad \sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L_2(\tau_k)} = \mathcal{O}(1),
$$

we obtain the estimates

$$
\begin{aligned}
\|\mathbf{u}_I-\mathbf{u}_h\|_1 &= \mathcal{O}(h) \\
\|p_I-p_h\|_1 &= \mathcal{O}(1).
\end{aligned}
\tag{34}
$$

Finally we invoke the triangle inequality to bound the discretization errors in H^1-norm

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_1 \le \|\mathbf{u}-\mathbf{u}_I\|_1 + \|\mathbf{u}_I-\mathbf{u}_h\|_1 = \\ = \mathcal{O}(h)+\mathcal{O}(h) = \mathcal{O}(h) \\ \|p-p_h\|_1 \le \|p-p_I\|_1 + \|p_I-p_h\|_1 = \\ = \mathcal{O}(h)+\mathcal{O}(1) = \mathcal{O}(1) \end{aligned} \tag{35}$$

Remark 4.2 *For some constant C, $\|\mathbf{u}-\mathbf{u}_h^*\|_1 \le C\|\mathbf{f}\|_0$ and $\sum_k \|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L_2(\tau_k)} \le C\|\mathbf{f}\|_0$. Then the estimates in Equation (35) read $\|\mathbf{u}-\mathbf{u}_h\|_1 \le Ch\|\mathbf{f}\|_0$ and $\|p-p_h\|_1 \le C\|\mathbf{f}\|_0$.*

5 Error bounds in L^2-norm

5.1 Displacement field

For the displacement field, consider the following dual problem to Equation (2):

$$\text{Find } \eta \in \mathbf{V} \text{ and } \xi \in P \text{ such that} \\ \left\{ \begin{array}{rcll} a(\alpha,\eta)+b(\alpha,\xi) & = & (\mathbf{u}-\mathbf{u}_h,\alpha) & \forall \alpha \in \mathbf{V} \\ b(\eta,\zeta)-c(\xi,\zeta) & = & 0 & \forall \zeta \in P \end{array} \right. . \tag{36}$$

If we make the special choice $\alpha = \mathbf{u}-\mathbf{u}_h$ we obtain,

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|_0 \quad = \quad & a(\mathbf{u}-\mathbf{u}_h,\eta)+b(\mathbf{u}-\mathbf{u}_h,\xi) = \\ & a(\mathbf{u}-\mathbf{u}_h,\eta)+b(\mathbf{u}-\mathbf{u}_h,\xi)\pm \\ & a(\mathbf{u}-\mathbf{u}_h,\eta_h)\pm b(\mathbf{u}-\mathbf{u}_h,\xi_h) = \\ & a(\mathbf{u}-\mathbf{u}_h,\eta-\eta_h,)+b(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)+ \\ & a(\mathbf{u}-\mathbf{u}_h,\eta_h)+b(\mathbf{u}-\mathbf{u}_h,\xi_h) \end{aligned} \tag{37}$$

The first two bilinear forms of the right hand side are bounded by

$$a(\mathbf{u}-\mathbf{u}_h,\eta-\eta_h) \quad \le \quad (2K_1+d\alpha+d\beta)\|\eta-\eta_h\|_1\|\mathbf{u}-\mathbf{u}_h\|_1 \tag{38}$$

$$b(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h) \quad \le \quad \mu\|\mathbf{u}-\mathbf{u}_h\|_1\|\xi-\xi_h\|_0. \tag{39}$$

To treat the third term in Equation (37), consider the first row of Equation (24) with $\mathbf{v}_h = \eta_h$

$$a(\mathbf{u}-\mathbf{u}_h,\eta_h)+b(\eta_h,p-p_h) = \mathbf{0}, \tag{40}$$

from which we derive the bound

$$\begin{aligned} a(\mathbf{u}-\mathbf{u}_h,\eta_h,) \quad & = \quad -b(\eta_h,p-p_h) \\ & \le \quad |b(\eta_h,p-p_h)| \\ & \le \quad \mu\|\eta_h\|_1\|p-p_h\|_0. \end{aligned} \tag{41}$$

The fourth term is slightly more involved. From the second row of

Equation (22) we have

$$
\begin{aligned}
b(\mathbf{u}-\mathbf{u}_h,\xi_h) &= c(p-p_h,\xi_h)+\sigma d(p-p_h,\xi_h)\\
&\quad \sigma e(\xi_h,\mathbf{u}-\mathbf{u}_h^*,\mathbf{b},\mathbf{c})\\
&\le \frac{\mu^2}{\lambda}\|p-p_h\|_0\|\xi_h\|_0+\sigma\mu\|p-p_h\|_1\|\xi_h\|_1\\
&\quad +\sigma 2\mu\|\xi_h\|_1\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}\\
&\quad +\sigma d\alpha\|\xi_h\|_1\|\mathbf{u}-\mathbf{u}_h^*\|_1+\sigma d\beta\|\xi_h\|_1\|\mathbf{u}-\mathbf{u}_h^*\|_1\\
&\le \frac{\mu^2}{\lambda}\|p-p_h\|_0 h|\xi_h|_1+\sigma\mu(1+h)\|p-p_h\|_1|\xi_h|_1\\
&\quad +\sigma 2\mu(1+h)|\xi_h|_1\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}\\
&\quad +\sigma d(\alpha+\beta)(1+h)|\xi_h|_1\|\mathbf{u}-\mathbf{u}_h^*\|_1,
\end{aligned}
\tag{42}
$$

where we use that $|\cdot|_1\le\|\cdot\|_1\le(1+h)|\cdot|_1$.

Summing up Equations (38) - (42) together yields

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0^2 &\le (2K_1+d\alpha+d\beta)\|\eta-\eta_h\|_1\|\mathbf{u}-\mathbf{u}_h\|_1\\
&\quad +\mu\|\mathbf{u}-\mathbf{u}_h\|_1\|\xi-\xi_h\|_0+\mu\|\eta_h\|_1\|p-p_h\|_0\\
&\quad +\frac{\mu^2}{\lambda}\|p-p_h\|_0 h|\xi_h|_1+\sigma\mu(1+h)\|p-p_h\|_1|\xi_h|_1\\
&\quad +\sigma 2\mu(1+h)|\xi_h|_1\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}\\
&\quad +\sigma d(\alpha+\beta)(1+h)|\xi_h|_1\|\mathbf{u}-\mathbf{u}_h^*\|_1,
\end{aligned}
\tag{43}
$$

In the following step, take the following relations into account,

- $|\xi|_1\le\|\mathbf{u}-\mathbf{u}_h$
- $|\eta|_2\le\|\mathbf{u}-\mathbf{u}_h$
- $\|\eta-\eta_h\|_1\le\|\eta-\eta_I\|_1+\|\eta_I-\eta_h\|_1\le Ch|\eta|_2+Ch\|\mathbf{u}-\mathbf{u}_h\|_0$
- $\|\xi-\xi_h\|_0\le\|\xi-\xi_I\|_0+\|\xi_I-\xi_h\|_0\le Ch|\xi|_2+Ch\|\mathbf{u}-\mathbf{u}_h\|_0$.
- Equation (43) holds for any ξ, and for ξ_I in particular. Hence, $\|\xi_h\|_0\to\|\xi_I\|_0\le h|\xi_I|_1\le Ch\|\mathbf{u}-\mathbf{u}_h\|_0$
- Equation (43) holds for any η, and for η_I in particular. $\|\eta_h\|_1\to\|\eta_I\|_1\le h|\eta_I|_2\le hC\|\mathbf{u}-\mathbf{u}_h\|_0$.

Substitute ξ_h by ξ_I and η_h by η_I, in Equation (43). After some manipulations we obtain

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0^2 &\le h(2K_1+d(\alpha+\beta)+\mu)\|\mathbf{u}-\mathbf{u}_h\|_1\|\mathbf{u}-\mathbf{u}_h\|_0\\
&\quad +h\mu\left(1+\frac{\mu}{\lambda}\right)\|\mathbf{u}-\mathbf{u}_h\|_0\|p-p_h\|_0\\
&\quad +\sigma\mu\|\mathbf{u}-\mathbf{u}_h\|_0\|p-p_h\|_1\\
&\quad +\sigma 2\mu(1+h)\|\mathbf{u}-\mathbf{u}_h\|_0\sum_k\|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)}\\
&\quad +\sigma d(\alpha+\beta)(1+h)\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{u}-\mathbf{u}_h^*\|_1,
\end{aligned}
\tag{44}
$$

which gives us the desired estimate

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0 \leq\ & h(2K_1 + d(\alpha+\beta)+\mu)\|\mathbf{u}-\mathbf{u}_h\|_1 \\
& + h\mu\left(1+\frac{\mu}{\lambda}\right)\|p-p_h\|_0 + \sigma\mu\|p-p_h\|_1 \\
& + \sigma 2\mu(1+h)\sum_k \|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)} \\
& + \sigma d(\alpha+\beta)(1+h)\|\mathbf{u}-\mathbf{u}_h^*\|_1.
\end{aligned}
\tag{45}
$$

That is,

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_0 \leq\ & hC'\|\mathbf{u}-\mathbf{u}_h\|_1 \\
& + hC''\|p-p_h\|_0 + \sigma C'''\mu\|p-p_h\|_1 \\
& + \sigma C''''\|\mathbf{f}\|_0,
\end{aligned}
\tag{46}
$$

for some constants C' - C'''' independent of h.

5.2 Pressure field

Consider now the following dual problem for the pressure field:

$$
\text{Find } \theta \in \mathbf{V} \text{ and } \chi \in P \text{ such that}
\quad
\left\{
\begin{array}{rcll}
a(\alpha,\theta)+b(\alpha,\chi) &=& \mathbf{0} & \forall \alpha \in \mathbf{V} \\
b(\theta,\zeta)-c(\chi,\zeta) &=& (p-p_h,\zeta) & \forall \zeta \in P
\end{array}
\right.
\tag{47}
$$

If we take $\zeta = p - p_h$, we find that

$$
\begin{array}{rcl}
\|p-p_h\|_0^2 &=& |b(\theta,p-p_h)-c(\chi,p-p_h)| \\
&=& |b(\theta-\theta_h,p-p_h)+b(\theta_h,p-p_h)-c(\chi,p-p_h)| \\
&=& |b(\theta-\theta_h,p-p_h)+b(\theta_h,p-p_h)-b(\theta,p-p_h)| \\
&=& |b(\theta-\theta_h,p-p_h)-b(\theta-\theta_h,p-p_h)| \\
&\leq& |b(\theta-\theta_h,p-p_h)|+|b(\theta-\theta_h,p-p_h)| \\
&\leq& 2\mu\|\theta-\theta_h\|_1\|p-p_h\|_0 \\
&\leq& 2\mu\|\theta-\theta_I\|_1\|p-p_h\|_0 + \|\theta_I-\theta_h\|_1\|p-p_h\|_0 \\
&\leq& 2\mu\|\theta-\theta_I\|_1\|p-p_h\|_1 + \|\theta_I-\theta_h\|_1\|p-p_h\|_1 \\
&\leq& Ch|\theta|_2\|p-p_h\|_1 + \|\theta_I-\theta_h\|_1\|p-p_h\|_1.
\end{array}
\tag{48}
$$

The same reasoning as for the displacement field applies, namely that $|\theta|_2 \leq \|p-p_h\|_0$ and $\|\theta_I-\theta_h\|_1 \leq h\|p-p_h\|_0$, which gives

$$
\|p-p_h\|_0 \leq h\|p-p_h\|_1 \tag{49}
$$

Combining the L^2-estimates Equation (45) and Equation (49) with the H^1-bounds for displacement and pressure in Equation (35), we get

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_0 + h\|p-p_h\|_0 \\
&\qquad \le hC_1'\|\mathbf{u}-\mathbf{u}_h\|_1 + hC_2'\|p-p_h\|_0 + \sigma C_3'\|p-p_h\|_1 \\
&\qquad + \sigma(1+h)C_4'\sum_k \|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)} \\
&\qquad + \sigma(1+h)C_5'\|\mathbf{u}-\mathbf{u}_h^*\|_1 \qquad (50)\\
&\qquad \le hC_1'\|\mathbf{u}-\mathbf{u}_h\|_1 + (h^2C_2'' + \sigma C_3')\|p-p_h\|_1 \\
&\qquad + \sigma(1+h)C_4'\sum_k \|\Delta(\mathbf{u}-\mathbf{u}_h^*)\|_{L^2(\tau_k)} \\
&\qquad + \sigma(1+h)C_5'\|\mathbf{u}-\mathbf{u}_h^*\|_1.
\end{aligned}
$$

Hence,

$$
\|\mathbf{u}-\mathbf{u}_h\|_0 + h\|p-p_h\|_0 \le h^2C'''(\|\mathbf{f}\| + \mathcal{O}(1)). \qquad (51)
$$

6 Conclusions

In this report we study a nonsymmetric mixed finite element problem for which the standard LBB condition does not hold. The problem is further consistently stabilized and the stability of the resulting system is analysed. It is shown that the choice of the stabilization parameter $\sigma = \mathcal{O}(h^2)$ ensures an optimal bound for the discretization error in the displacements $\mathbf{u}$ and in the pressure p.

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