Boundary Conditions for a Divergence Free Velocity-Pressure Formulation of the Incompressible Navier-Stokes Equations

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Abstract

New sets of boundary conditions for the velocity-pressure formulation of the incompressible Navier-Stokes equations are derived. The boundary conditions have the same form on both inflow and outflow boundaries and lead to a divergence free solution. Moreover, the specific form of the boundary conditions makes it possible derive a symmetric positive definite equation system for the internal pressure. Numerical experiments support the theoretical conclusions.

1 Introduction

In solving the incompressible Navier-Stokes equations both the velocity-divergence and the velocity-pressure forms are considered. With the velocity-divergence form, careful consideration must be given to the discretization of the continuity equation, in order to prevent odd-even point decoupling and also to maintain the divergence free condition. Alternative methods for this form such as the vorticity/stream function formulation and the staggered grid approach are frequently applied in two dimensions. Both of these methods become less attractive in three dimensions, due to either boundary condition considerations or computational complexity.

To avoid the difficulties mentioned above and to increase the potential for computational efficiency when solving general problems, the velocity-pressure form of the equations is chosen. The main drawback with this form is that the direct enforcement of zero divergence is lost. There has been considerable discussion in the literature regarding the proper boundary

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condition for the pressure equation. According to [1] the normal component of the momentum equation is the appropriate boundary condition. However, as indicated in [2],[3] this particular boundary condition allows a nonzero divergence on the boundary, and thus, allows a nonzero divergence in the domain of interest. Furthermore, in [4] it is pointed out that this boundary condition provides no new information, resulting in an under-determined system. In [5] the author considers the solid boundary condition for the pressure equation and imposes the divergence or the normal derivative of the divergence in combination with the normal component of the momentum equation. This work is related to the plan of attack in that paper.

Staggered grids are often used in the numerical solution of the incompressible Navier-Stokes equations. That problem which more or less excludes the use of the energy method as an analysis tool is considered in [6],[7],[8],[9]. Another way of dealing with this problem is to use projection or fractional step methods, see [10],[11],[12]. The solution is computed in a sequence. Typically, one first computes an approximation of the velocity field that do not satisfy the divergence relation and where no pressure is involved. Next, an approximation of the pressure is computed that leads eventually leads to a correction of the velocity field that satisfy the divergence relation. These methods are very popular but in most case low order accurate (due to the sequential computation). We will not consider them in this paper.

Instead we will revisit the problem with switching from the velocity-divergence to the velocity-pressure form and derive necessary and sufficient conditions for obtaining the same solution, see also [13]. Our conditions are derived in the continuous setting and therefore of general use (any numerical method can be used). The paper is organized in the following way. In section 2, we analyze the nonlinear problem and derive new boundary conditions. The results from the nonlinear problem are applied to an example in section 3. The discrete problem is analyzed in section 4. Numerical calculations are performed in section 5. Conclusions are drawn in section 6.

2 Analysis of the nonlinear problem

The initial-boundary-value problem for the two-dimensional incompressible Navier-Stokes equations on the conventional velocity-divergence form can be written as

$$\begin{aligned} \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla p - \nu\Delta\mathbf{V} &= \mathbf{f}, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ \nabla\cdot\mathbf{V} &= 0, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ B(\mathbf{V},p) &= \mathbf{g}, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0 \\ (\mathbf{V},p) &= \mathbf{h}, \qquad \mathbf{x}\in\Omega, \quad t=0 \end{aligned} \tag{1}$$

where $\mathbf{V} = (u,v)^T$ is the fluid velocity, p is the pressure, $\mathbf{f}$ is the forcing function, $\mathbf{g}$ the boundary data and $\mathbf{h}$ the initial data. ∇ and Δ are the gradient and Laplacian operators respectively. The density is taken to be one, and the coefficient ν is the reciprocal of the Reynolds number, $\nu = 1/Re$. The domain Ω is in $\Re^2$, and the boundary of the domain is $\partial\Omega$. The two boundary conditions are represented by the boundary operator $B(\mathbf{V},p)$ and the boundary data $\mathbf{g}$.

There are alternative forms for the system of Eq. (1). One of these forms can be derived by taking the divergence of the momentum equation and applying $\nabla \cdot \mathbf{V} = 0$. The resulting system is is called the velocity-pressure (or pressure Poisson) form of the equations, and the associated initial-boundary-value problem is defined by

$$\begin{aligned} \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla p - \nu\Delta\mathbf{V} &= \mathbf{f}, && \mathbf{x}\in\Omega, \quad t \geq 0 \\ \Delta p &= \tilde{\mathbf{f}}, && \mathbf{x}\in\Omega, \quad t \geq 0 \\ B(\mathbf{V}, p) &= \mathbf{g}, && \mathbf{x}\in\partial\Omega, \quad t \geq 0 \\ \tilde{B}(\mathbf{V}, p) &= \tilde{\mathbf{g}}, && \mathbf{x}\in\partial\Omega, \quad t \geq 0 \\ (\mathbf{V}, p) &= \mathbf{h}, && \mathbf{x}\in\Omega, \quad t = 0 \end{aligned} \tag{2}$$

where $\tilde{\mathbf{f}} = (\nabla \cdot \mathbf{f} - (\nabla u \cdot \mathbf{V}_x + \nabla v \cdot \mathbf{V}_y)$. Now, the pressure equation plays the role of the continuity equation. Since this equation is second order, the system requires an additional boundary condition (with boundary operator $\tilde{B}$).

Note that even though one can derive the pressure Poisson equation by assuming zero divergence, it is not clear that the implication goes the other way. In other words, the simple replacement of divergence relation with the pressure equation *does not* imply zero divergence. Other elements must also be considered. The main part of this paper is devoted to a step by step derivation (using classical initial boundary value theory) of why (2) is an appropriate form of (1) and what kind of boundary conditions ($\tilde{B}$ and $\tilde{\mathbf{g}}$) and initial conditions ($\mathbf{h}$) the system (2) requires in order to produce identically the same solution as the one given by (1).

2.1 Replacing the divergence condition

Let us start by considering the well posedness of a time-dependent scalar advection-diffusion equation with a general set of data,

$$\begin{aligned} \phi_t + (\vec{U}\cdot\nabla)\phi - \nu\Delta\phi &= \mathbf{F}, \quad \mathbf{x}\in\Omega \quad t \geq 0 \\ L\phi &= \mathbf{G}, \quad \mathbf{x}\in\partial\Omega \quad t \geq 0 \\ \phi &= \mathbf{H}, \quad \mathbf{x}\in\Omega \quad t = 0. \end{aligned} \tag{3}$$

The data in (3) are the forcing function $\mathbf{F}$, the boundary data $\mathbf{G}$ and the initial data $\mathbf{H}$. For reasons that become evident later we require $\mathbf{H}$ to be small. Also unknown in (3) is the boundary operator L. For the variable coefficient case, $\vec{U} = (a, b)$ vary in space and time while for the non-linear case we have $\vec{U} = (u, v)$ and $\phi = \nabla \cdot \vec{U}$. The viscosity $\nu << 1$ is constant and small.

Remark: Consider the problem (1). By taking the divergence of the momentum equation and *not* imposing the divergence condition we obtain the advection-diffusion problem (3) where $\phi = \nabla \cdot \mathbf{V}$, $\vec{U} = \mathbf{V}$ and $\mathbf{F} = \tilde{\mathbf{f}} - \Delta p$. However, and we stress this point, we will first consider (3) as a fully general advection-diffusion problem and determine general conditions that leads to $\phi = 0$. Once that is done, we can relate that general result to the specific Navier-Stokes formulation (2) and explicitly state the conditions that are required for obtaining zero divergence. This is the main theme of the paper.

The energy method (multiply with ϕ and integrate over the domain) applied to (3) yields,

$$||\phi||_t^2 + 2\nu \int_\Omega |\nabla\phi|^2 dxdy = -\oint_{\partial\Omega} (\vec{U}\cdot\vec{n})\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} ds + \int_\Omega (\nabla\cdot\vec{U})\phi^2 + 2\phi\mathbf{F} dxdy, \quad (4)$$

where $\vec{n}$ is the outward pointing normal. Clearly we need to bound the two terms on the righthand side of (4).

Let us start with the boundary integral and assume that $U_n = \vec{U}\cdot\vec{n} \neq 0$. We rewrite the boundary terms as,

$$U_n\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} = U_n^{-1}[(U_n\phi - \nu\frac{\partial\phi}{\partial n})^2 - (\nu\frac{\partial\phi}{\partial n})^2]. \quad (5)$$

A boundary condition that cancels the quadratic form with the wrong sign in (4) is

$$\tilde{B}(\mathbf{V}, p) = L\phi = \frac{(U_n - |U_n|)}{2}\phi - \nu\frac{\partial\phi}{\partial n} = \mathbf{G}. \quad (6)$$

The boundary condition (6) yields the estimate

$$-\oint_{\partial\Omega} U_n\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} ds \leq \oint_{\partial\Omega} \frac{1}{|U_n|}(\mathbf{G}^2 - (U_n\phi - \mathbf{G})^2) ds, \quad (7)$$

both for inflow ($U_n < 0$) and outflow ($U_n > 0$).

Remark: Other possible boundary conditions (that include the case where $U_n = 0$) except (6) that can be used are $\phi = 0$ both for inflow and outflow and $\partial\phi/\partial n = 0$ for outflow. However, (6) is the most dissipative condition.

Next we consider the second potential growth term in (4). We get the estimate

$$\int_\Omega (\nabla\cdot\vec{U})\phi^2 + 2\phi\mathbf{F} dxdy \leq (\eta + \delta)||\phi||^2 + \frac{1}{\eta}||\mathbf{F}||^2. \quad (8)$$

For the linear problem we have $\delta = |\nabla\cdot\vec{U}|_{max}$. To get an estimate in the nonlinear problem we consider short times $t \leq T_0$ and recall that $\phi(x, 0) = \mathbf{H}$ where $\mathbf{H}$ is small. Note also that ϕ obeys a maximum principle (the magnitude of local maxima and minima decrease). This implies that ϕ is limited for short times and consequently we can choose $\delta = |\phi|_{max}$ for $t \leq T_0$ and $\mathbf{x} \in \partial\Omega$.

By defining the boundary norm as $||\mathbf{U}||_{\partial\Omega}^2 = \oint_{\partial\Omega} \mathbf{U}^2/|U_n| ds$, introducing the notation $\tilde{\eta} = \eta + \delta$ and inserting (7),(8) into (4) we obtain the final estimate

$$||\phi||^2 + \int_0^T (2\nu||\nabla\phi||^2 + ||U_n\phi - \mathbf{G}||_{\partial\Omega}^2) e^{-\tilde{\eta}(t-T)} dt \leq ||\mathbf{H}||^2 e^{\tilde{\eta}T} + \int_0^T (\frac{1}{\tilde{\eta}}||\mathbf{F}||^2 + ||\mathbf{G}||_{\partial\Omega}^2) e^{-\tilde{\eta}(t-T)} dt. \quad (9)$$

In the nonlinear case, (9) is valid for $T \leq T_0$.

We are now ready to prove the following proposition.

Proposition 2.1 *Consider the problem (3). If the boundary conditions are given by (6) and*

$$\mathbf{F} = 0, \quad \mathbf{G} = 0, \quad \mathbf{H} = 0, \tag{10}$$

then the L_2 norm of the divergence $||\phi||^2$ will be zero for all times.

Proof: The conditions (10) inserted into (9) yields

$$||\phi||^2 + \int_0^T (2\nu||\nabla\phi||^2 + ||U_n\phi||^2_{\partial\Omega})e^{-\tilde{\eta}(t-T)}dt \leq 0 \tag{11}$$

In the linear case this concludes the proof. In the nonlinear case, (11) is valid for small times $T \leq T_0$ and we obtain $||\phi||^2(T_0) = 0$. A repetition of the argument proves proposition 2.1 for the nonlinear case for all times $\blacksquare$.

Remark: Note that for long time calculations, $\mathbf{H} \neq 0$ but sufficiently small might suffice since initial non-zero divergence effects will might decay with time (at least with (6) as the boundary condition), see equation (7),(8) and (4).

2.2 The energy estimate for the momentum equations

Multiplication of the top two equations in (2) with $\mathbf{V}$ and integration over the domain yields,

$$||\mathbf{V}||_t^2 + 2\nu(||\nabla u||^2 + ||\nabla v||^2) = -\oint_{\partial\Omega} W^T A(\vec{U})W ds + 2\int_\Omega \phi(p + \mathbf{V}^2/2) + \mathbf{V}\cdot\mathbf{f}dxdy, \tag{12}$$

where $\phi = \nabla \cdot \mathbf{V}$ and

$$W = \begin{pmatrix} u_n \\ u_t \\ p \\ \nu(\partial u_n/\partial n) \\ \nu(\partial u_t/\partial n) \end{pmatrix}, \quad A(\mathbf{V}) = \begin{pmatrix} U_n & 0 & +1 & -1 & 0 \\ 0 & U_n & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

In (12),(13) u_n, u_t denotes the normal velocity and tangential velocity and $U_n = \mathbf{V}\cdot\vec{n}$. By choosing the boundary conditions and initial data discussed in the previous section we have $\phi = 0$.

To bound the energy in (12) we need to make sure that $W^T AW \leq 0$. The correct number of boundary conditions and a maximally dissipative form of the boundary conditions are obtained by specifying the correct characteristic variable, for more details see [14]. We can write the condition on diagonal form as

$$W^T AW = (X^T W)^T \Lambda (X^T W)^T$$

and specify $(X^T W)_i = C_i$ whenever $\lambda_i < 0$. Straightforward but tedious algebra yields,

$$\lambda_{1,2} = \frac{U_n}{2} \pm \sqrt{(\frac{U_n}{2})^2 + 1}, \quad \lambda_3 = 0, \quad \lambda_{4,5} = \frac{U_n}{2} \pm \sqrt{(\frac{U_n}{2})^2 + 2}, \tag{14}$$

$$X^T = \begin{bmatrix} \sqrt{1+\lambda_1^2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1+\lambda_2^2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & & 0 \\ 0 & 0 & 0 & \sqrt{2+\lambda_4^2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2+\lambda_5^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & -1 \\ 0 & \lambda_2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ \lambda_4 & 0 & 1 & -1 & 0 \\ \lambda_5 & 0 & 1 & -1 & 0 \end{bmatrix}. \quad (15)$$

Note that λ_2, λ_5 are negative for all values of U_n. This implies that the conditions

$$C_2 = \lambda_2 u_t - \nu \frac{\partial u_t}{\partial n} = g_2, \quad C_5 = \lambda_5 u_n - \nu \frac{\partial u_n}{\partial n} + p = g_5, \quad (16)$$

lead to well posedness. Our result can be formulated as

$$B(\mathbf{V}, p) = \begin{pmatrix} 0 & \lambda_2 - \nu\partial/\partial n & 0 \\ \lambda_5 - \nu\partial/\partial n & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ u_t \\ p \end{pmatrix} = \mathbf{g}. \quad (17)$$

Remark: Note that (16),(17) can be imposed on both inflow and outflow boundaries which simplifies numerical calculations since it is not necessary to switch type of boundary conditions. For other types of boundary conditions that lead to well-posedness, see [15].

By choosing the boundary conditions (17) we obtain the final estimate

$$||\mathbf{V}||^2 + \int_0^T 2\nu(||\nabla u||^2 + ||\nabla v||^2)e^{-\eta(t-T)} + \int_0^T ||C^+||_\Gamma^2 e^{-\eta(t-T)} dt \leq$$
$$||\mathbf{h}||^2 e^{\eta T} + \int_0^T (\frac{1}{\eta}||\mathbf{f}||^2 + ||\mathbf{g}||_\Gamma^2) e^{-\eta(t-T)} dt. \quad (18)$$

where $||C^+||_\Gamma^2 = \sum_{i=1,3,4} \oint_\Gamma \lambda_i C_i^2 ds$, $||\mathbf{g}||_\Gamma^2 = \sum_{i=2,5} \oint_\Gamma |\lambda_i| g_i^2 ds$ and $\Gamma = \partial\Omega$.

2.3 The velocity-pressure formulation for zero divergence

The main result in this paper is

Proposition 2.2 *Consider the velocity-pressure formulation (2). If the boundary conditions are given by (6) and (17) with $\tilde{\mathbf{g}} = 0$ and $\nabla \cdot \mathbf{h} = 0$. Then (2) is strongly well-posed and has a divergence free solution that satisfies the estimate (18).*

Proof: See section 2.1-2.2 above. ■

A similar result was obtained in [16] by choosing the boundary condition for the advection-diffusion problem (3) to be $\nabla \cdot \mathbf{V} = 0$. We can summarize the result in proposition 2.1-2.3 in the following way. By

- solving Poisson's equation for the pressure ($\mathbf{F} = \tilde{\mathbf{f}} - \Delta p = 0$),
- using dissipative ($B(\mathbf{V}, p)$) boundary conditions for the momentum equations,

- using dissipative ($\tilde{B}(\mathbf{V}, p)$) boundary conditions for the divergence equation,
- using zero boundary data ($\tilde{\mathbf{g}} = 0$) for the divergence boundary conditions,
- initializing with a divergence free velocity field ($\nabla \cdot \mathbf{h} = 0$),

a strongly well-posed divergence-free solution to the velocity-pressure formulation (2) will be obtained.

3 An example

In this section we will investigate the new boundary conditions in some detail by using the Laplace-Fourier transform technique on a constant coefficient version of problem (1) and (2). By Laplace-Fourier tranforming these problems, we get two systems of ordinary differential equations of the form

$$\mathbf{A}V_{xx} + \mathbf{B}V_x + \mathbf{C}V = \tilde{\mathbf{F}}, \quad x \geq 0. \tag{19}$$

In (19), $\tilde{\mathbf{F}}$ includes the Laplace-Fourier transformed forcing functions and initial data present in (1) and (2). V is the Laplace-Fourier transform of $(\mathbf{V}, p)^T$. The constant matrices in (19) are

$$\mathbf{A} = \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \tilde{s} & 0 & 0 \\ 0 & \tilde{s} & i\omega \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{20}$$

$$\mathbf{A} = \begin{pmatrix} -\nu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \tilde{s} & 0 & 0 \\ 0 & \tilde{s} & i\omega \\ 0 & i\omega & 0 \end{pmatrix}, \tag{21}$$

for the velocity-pressure and velocity-divergence form respectively. The constants and parameters $\bar{u}, \bar{v}, \nu, s, i\omega$ in (20) and (21) are the velocity components in x, y direction, the viscosity, the dual variable to time and the dual variable to y respectively. We have also used the abbreviation $\tilde{s} = s + i\omega\bar{v} + \nu\omega^2$.

We limit the analysis to the case where $|\omega| \geq \omega_0 > 0$ since the equations collapse for $\omega = 0$. This means that we exclude solutions with zero mean value in the y direction. The top two rows are identical for both formulations and represent the momentum equations. The bottom row represents the Poisson equation for the pressure and the divergence relation respectively.

The system (19) defined in the half space $x \geq 0$ must be augmented with boundary conditions. At infinity ($x \to \infty$) we demand a bounded solution which implies that only decaying modes will be considered. At $x = 0$ we impose boundary conditions.

The solution is a sum of the homogeneous and particular solution, i.e. $V = U + U_p$. The particular solution U_p is a smooth bounded function of the data of the problem, i.e. $U_p = U_p(\tilde{\mathbf{F}})$. We assume that the data have compact support in $x \geq 0$, i.e. that $\tilde{\mathbf{F}}(|\mathbf{x}| > R) = 0$. The homogeneous solution is obtained by making the ansatz $U = \phi \exp{(\kappa x)}$ and inserting that into the homogeneous version of (19). That gives us the generalized eigenvalues and eigenvectors as solutions to

$$|\mathbf{A}\kappa_i^2 + \mathbf{B}\kappa_i + \mathbf{C}| = 0, \quad (\mathbf{A}\kappa_i^2 + \mathbf{B}\kappa_i + \mathbf{C})\phi_i = 0, \tag{22}$$

where $i = 1 - 6$ for the velocity-pressure case and $i = 1 - 4$ for the velocity-divergence case.

3.1 The velocity-pressure formulation

In the velocity-pressure case we get

$$\kappa_{1,2} = \pm|\omega|, \quad \kappa_{34,56} = (\frac{\bar{u}}{2\nu}) \pm \sqrt{(\frac{\bar{u}}{2\nu})^2 + \frac{\tilde{s}}{\nu}}. \tag{23}$$

The double roots $\kappa_{34,56}$ indicates that the proper ansatz is $U = (\phi_0 + x\phi_1)\exp(\kappa_{34,56}x)$. By using that we get $\phi_0 = 0$ and the solution

$$U = \sum_{i=1}^{6} \sigma_i \psi_i e^{\kappa_i x}, \tag{24}$$

where the $\sigma_i : s$ are constant to be determined by the boundary conditions and

$$\psi_1 = \begin{bmatrix} 1 \\ +\frac{\omega}{|\omega|} \\ -\frac{\beta_1}{|\omega|} \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ -\frac{\omega}{|\omega|} \\ +\frac{\beta_2}{|\omega|} \end{bmatrix}, \quad \psi_3 = \psi_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \psi_4 = \psi_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{25}$$

The parameters in (25) are defined as $\beta_1 = \tilde{s} + \bar{u}|\omega| - \nu|\omega|^2$, $\beta_2 = \tilde{s} - \bar{u}|\omega| - \nu|\omega|^2$. A decaying solution for a positive real part in s and positive x is

$$U = \sigma_2\psi_2 e^{\kappa_2 x} + (\sigma_5\psi_5 + \sigma_6\psi_6)e^{\kappa_5 x}. \tag{26}$$

There is a possibility of a triple root case (for a more detailed discussion on multiple roots, see [15]) since the parameters β_1, β_2 might vanish. For that case we have

$$\kappa_{34} = \kappa_1 + \sqrt{(\frac{\bar{u}}{2\nu} - |\omega|)^2 + \frac{\beta_1}{\nu}} - (\frac{\bar{u}}{2\nu} - |\omega|), \quad \kappa_{56} = \kappa_2 + (\frac{\bar{u}}{2\nu} + |\omega|) - \sqrt{(\frac{\bar{u}}{2\nu} + |\omega|)^2 + \frac{\beta_2}{\nu}}. \tag{27}$$

The triple root ansatz $U = (\phi_0 + x\phi_1 + x^2\phi_2)\exp(\kappa_2 x)$ yields the decaying solution

$$U = \left[\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + xc_3 \begin{bmatrix} \frac{|\omega|}{\bar{u}+2\nu|\omega|} \\ -\frac{i\omega}{\bar{u}+2\nu|\omega|} \\ 0 \end{bmatrix}\right] e^{-|\omega|x}. \tag{28}$$

By writing the decaying double root solution (26) in a modified way as

$$U = \frac{\sigma_2}{\kappa_5 - \kappa_2}\psi_2 e^{\kappa_2 x} + \left[\begin{bmatrix} \sigma_5 \\ \sigma_6 \\ 0 \end{bmatrix} - \frac{\sigma_2}{\kappa_5 - \kappa_2}\begin{bmatrix} 1 \\ -\frac{i\omega}{|\omega|} \\ 0 \end{bmatrix}\right] e^{\kappa_5 x}, \tag{29}$$

using (27), making the substitutions $c_1 = \sigma_5, c_2 = \sigma_6$ and $c_3 = -\sigma_2\frac{\bar{u}+2\nu|\omega|}{|\omega|}$ and letting $\beta_2 \to 0$ we find that (29) converge to (28).

3.2 The velocity-divergence formulation

Essentially the same procedure as described above yields the four generalized eigenvalues

$$\kappa_{1,2} = \pm|\omega|, \quad \kappa_{3,5} = (\frac{\bar{u}}{2\nu}) \pm \sqrt{(\frac{\bar{u}}{2\nu})^2 + \frac{\tilde{s}}{\nu}}. \tag{30}$$

The numbering in (30) is chosen to simplify the comparison with the velocity-pressure formulation above. Without going into detail, we state that the decaying single root solution that converges to the double root solution is

$$U = \frac{\alpha_2}{\kappa_5 - \kappa_2}\psi_2 e^{\kappa_2 x} + (\alpha_5 - \frac{\alpha_2}{\kappa_5 - \kappa_2}) \begin{bmatrix} 1 \\ \frac{i\kappa_5}{\omega} \\ 0 \end{bmatrix} e^{\kappa_5 x}. \tag{31}$$

Again we use the notation established in the previous section.

3.3 The effect of the new boundary conditions

Now we will investigate what effect the divergence boundary condition (6) has on the velocity-pressure solution. We can prove

Proposition 3.1 *Consider the velocity-pressure formulation (29). By applying the divergence boundary condition (6) we obtain the divergence free solution given in (31).*

Proof: The boundary condition (6) applied on the boundary $x = 0$ with the outward normal pointing in the negative x direction and the velocity $\mathbf{V} = (\bar{u}, \bar{v})^T$ is

$$(\frac{\bar{u} + |\bar{u}|}{2})\phi - \nu\phi_x = 0. \tag{32}$$

With $\bar{u} > 0$ we have inflow and a Robin boundary condition while $\bar{u} < 0$ yields a clean Neumann condition. Both cases can be treated simultaneously, and (32) applied to (29) yields

$$\sigma_6 = \frac{1}{i\omega}\sigma_2 - \frac{\kappa_5}{i\omega}\sigma_5,$$

which inserted in (29) yields (31) exactly. ■

An almost immediate consequence of Proposition 3.3 is

Lemma 3.1 *The divergence free form of the velocity-pressure formulation augmented with the boundary conditions (17) yields a pointwise bounded solution.*

Proof: The boundary conditions (17) leads to an energy estimate of the form (18) which bounds the velocity components in Laplace-Fourier space pointwise. By normalizing the eigenvectors in (31) with $|\psi_2|$ we get

$$U = \frac{\tilde{\alpha}_2}{\kappa_5 - \kappa_2}\frac{1}{|\psi_2|}\psi_2 e^{\kappa_2 x} + (\tilde{\alpha}_5 - \frac{\tilde{\alpha}_2}{\kappa_5 - \kappa_2})\frac{1}{|\psi_2|} \begin{bmatrix} 1 \\ \frac{i\kappa_5}{\omega} \\ 0 \end{bmatrix} e^{\kappa_5 x}. \tag{33}$$

The boundedness of the velocity components now implies that the coefficients $\tilde{\alpha}_2, \tilde{\alpha}_5$ are bounded. This in turn implies that the pressure is also bounded since

$$p = \tilde{\alpha}_2 \frac{\beta_2}{(\kappa_5 - \kappa_2)|\omega|} \frac{1}{|\psi_2|} e^{\kappa_2 x},$$

and $|\omega| > \omega_0 > 0$. The inverse Laplace-Fourier transform yields the same result in the non-transformed space ■.

4 The numerical approximation

The result of the analysis will be evaluated by using a high order finite difference technique. To construct the scheme we use one-dimensional summation-by-parts (SBP) operators. To make the paper self contained we start of with a short description (for more details see [17],[18],[19]). We define the first and second derivative operators of interest in this paper.

Definition 4.1 *A difference operator $D_1 = H^{-1}Q$ approximating $\partial/\partial x$ is a first derivative SBP operator if $H = H^T > 0$ and $Q + Q^T = B = diag\,(-1, 0 \ldots, 0, 1)$.*

Definition 4.2 *A difference operator $D_2 = H^{-1}(-M + BS)$ approximating $\partial^2/\partial x^2$ is said to be a symmetric second derivative SBP operator if $M = M^T \geq 0$, if S includes an approximation of the first derivative operator at the boundary and $B = diag\,(-1, 0 \ldots, 0, 1)$.*

The first an second derivative operators in this paper are based on diagonal norms. They are half as accurate at the boundaries as in the interior. The theory in [20] implies that first derivative operators of order 4,6,8 are lead to globally 3,4,5th order accurate schemes for hyperbolic problems and 4,5,6 th order global accuracy for parabolic problems.

We consider a two-dimensional domain with an $N+1 \times M+1$-points equidistant grid. The numerical approximation at grid point (x_i, y_j) is denoted $v_{i,j}$. We define a discrete solution vector $v^T = [v^0, v^1, \ldots, v^N]$, where $v^k = [v_{k,0}, v_{k,1}, \ldots, v_{k,M}]$ is the solution vector at x_k along the y-direction, as illustrated in Figure 1. To simplify the notation we introduce $v_{w,e,s,n}$, to define the boundary values at the west, east, south and north boundaries (see Figure 1). To distinguish if a difference operator D is operating in the x or y-directions we use the notations D_x and D_y. The following 2-dimensional operators

$$\begin{array}{llll} D_x & = (D_1 \otimes I_y), & D_y & = (I_x \otimes D_1) \\ D_{2x} & = (D_2 \otimes I_y), & D_{2y} & = (I_x \otimes D_2) \\ H_x & = (H \otimes I_y), & H_y & = (I_x \otimes H) \end{array}, \qquad (34)$$

will be frequently used. D_1, D_2 and H are the one dimensional operators discussed above and $I_{x,y}$ are the identity matrices. All matrices have the appropriate size in the x and y direction respectively. We have introduced the Kronecker product

$$C \otimes D = \begin{bmatrix} c_{0,0}\, D & \cdots & c_{0,q-1}\, D \\ \vdots & & \vdots \\ c_{p-1,0}\, D & \cdots & c_{p-1,q-1}\, D \end{bmatrix},$$

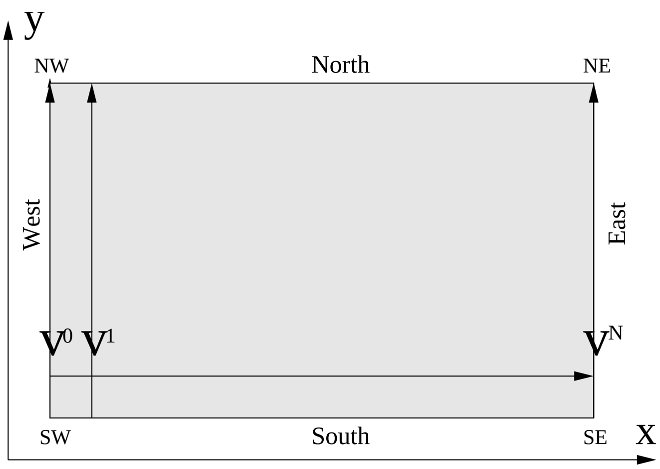


Figure 1: Two-dimensional domain showing the orientation of the solution vectors

where C is a $p \times q$ matrix and D is a $m \times n$ matrix. Two useful rules for the Kronecker product are $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ and $(A \otimes B)^T = A^T \otimes B^T$.

4.1 The semi-discrete approximation

The semi-discrete finite difference approximation (including a skew-symmetric splitting of the convective terms) of the momentum equations can be written

$$\begin{aligned} u_t + D_x^{(skew)} u + D_y^{(skew)} u + D_x p &= \nu D_L u + SAT^{(u)} + F^{(u)} \\ v_t + D_x^{(skew)} v + D_y^{(skew)} v + D_y p &= \nu D_L v + SAT^{(v)} + F^{(v)}. \end{aligned} \tag{35}$$

In (35), $F^{(u,v)}$ are the discrete forcing functions, and $D_{x,y}^{skew}$ the skew-symmetric splitted operator in the x and y direction respectively,

$$D_x^{(skew)} \equiv \frac{1}{2} A D_x + \frac{1}{2} D_x A - \frac{1}{2} A_x, \quad D_y^{(skew)} \equiv \frac{1}{2} B D_y + \frac{1}{2} D_y B - \frac{1}{2} B_y.$$

The matrices A, B, A_x, B_y have the values of $u, v, D_x u, D_y v$ injected on the diagonal. We have also introduced $D_L = D_{2x} + D_{2y}$ which is the discrete Laplacian. The boundary conditions are introduced as penalty terms $SAT^{(u,v)}$, using the SAT method, see for example [21],[22] and [19]. The discrete Poisson system for the pressure, see equation (2), is given by

$$D_L p = f \ , \tag{36}$$

where $f = -\left((D_x u)(D_x u) + 2(D_y u)(D_x v) + (D_y v)(D_y v)\right) + D_x F^{(u)} + D_y F^{(v)}$.

4.2 Boundary conditions for the semi-discrete approximation

The discrete version of the boundary condition (6) that lead to maximum dissipation of the divergence $u_x + v_y = \phi$ is given by

$$\begin{array}{lll} L_w^{(\phi)} & = \frac{(u+|u|)}{2}(D_x u + D_y v)_w - \nu(D_x(D_x u + D_y v))_w & = 0 \\ L_e^{(\phi)} & = \frac{(u-|u|)}{2}(D_x u + D_y v)_e - \nu(D_x(D_x u + D_y v))_e & = 0 \\ L_s^{(\phi)} & = \frac{(v+|v|)}{2}(D_x u + D_y v)_s - \nu(D_y(D_x u + D_y v))_s & = 0 \\ L_n^{(\phi)} & = \frac{(v-|v|)}{2}(D_x u + D_y v)_n - \nu(D_y(D_x u + D_y v))_n & = 0 \end{array}. \tag{37}$$

Remark: Other possible boundary conditions except (37) that can be used are $(D_x u + D_y v) = 0$ both for inflow and outflow and $\partial(D_x u + D_y v)\partial n = 0$ for outflow.

The discrete boundary conditions for the x-component u, at the boundaries are given by

$$\begin{array}{lll} L_w^{(u)} & = -\lambda_5^{(w)} u_w - \nu(Su)_w + p_w & = g_w^{(u)} \\ L_e^{(u)} & = +\lambda_5^{(e)} u_e - \nu(Su)_e + p_e & = g_e^{(u)} \\ L_s^{(u)} & = -\lambda_2^{(s)} u_s - \nu(Su)_s & = g_s^{(u)} \\ L_n^{(u)} & = +\lambda_2^{(n)} u_n - \nu(Su)_n & = g_n^{(u)} \end{array}. \tag{38}$$

The corresponding boundary conditions for the y-component v are given by

$$\begin{array}{lll} L_w^{(v)} & = -\lambda_2^{(w)} v_w - \nu(Sv)_w & = g_w^{(v)} \\ L_e^{(v)} & = +\lambda_2^{(e)} v_e - \nu(Sv)_e & = g_e^{(v)} \\ L_s^{(v)} & = -\lambda_5^{(w)} v_s - \nu(Sv)_s + p_s & = g_s^{(v)} \\ L_n^{(v)} & = +\lambda_5^{(e)} v_n - \nu(Sv)_n + p_n & = g_n^{(v)} \end{array}. \tag{39}$$

The expressions for the two eigenvalues $\lambda_{2,5}$ are given by (14) at each boundary, we get,

$$\begin{array}{ll} \lambda_5^{(w)} = \frac{-u_w}{2} - \sqrt{\left(\frac{-u_w}{2}\right)^2 + 2} & , \quad \lambda_2^{(w)} = \frac{-u_w}{2} - \sqrt{\left(\frac{u_w}{2}\right)^2 + 1} \\ \lambda_5^{(e)} = \frac{u_e}{2} - \sqrt{\left(\frac{u_e}{2}\right)^2 + 2} & , \quad \lambda_2^{(e)} = \frac{u_e}{2} - \sqrt{\left(\frac{u_e}{2}\right)^2 + 1} \\ \lambda_5^{(s)} = \frac{-v_s}{2} - \sqrt{\left(\frac{-v_s}{2}\right)^2 + 2} & , \quad \lambda_2^{(s)} = \frac{-v_s}{2} - \sqrt{\left(\frac{-v_s}{2}\right)^2 + 1} \\ \lambda_5^{(n)} = \frac{v_n}{2} - \sqrt{\left(\frac{v_n}{2}\right)^2 + 2} & , \quad \lambda_2^{(n)} = \frac{v_n}{2} - \sqrt{\left(\frac{v_n}{2}\right)^2 + 1} \end{array}. \tag{40}$$

The penalty terms for the x- and y- component (u, v) are given by

$$\begin{aligned} SAT^{(u,v)} = & \tau_w^{(u,v)} H_x^{-1} e_0 \otimes \left(L_w^{(u,v)} - g_w^{(u,v)}\right) + \tau_e^{(u,v)} H_x^{-1} e_N \otimes \left(L_e^{(u,v)} - g_e^{(u,v)}\right) \\ & + \tau_s^{(u,v)} H_y^{-1} \left(L_s^{(u,v)} - g_s^{(u,v)}\right) \otimes e_0 + \tau_n^{(u,v)} H_y^{-1} \left(L_n^{(u,v)} - g_n^{(u,v)}\right) \otimes e_N, \end{aligned}$$

where $e_0 = [1, 0, ..., 0]^T, e_N = [0, ..., 0, 1]^T$. The eight penalty parameters, $\tau_{w,e,s,n}^{(u,v)}$ will be tuned to obtain and energy estimate.

By multiplying the first and the second equation in (35) by $u^T\bar{H}$ and $v^T\bar{H}$ respectively and adding the transpose we obtain,

$$\left(\|u\|_{\bar{H}}^2 + \|v\|_{\bar{H}}^2\right)_t = BT + DI + RT. \tag{41}$$

In (41), $\bar{H} \equiv H_x H_y$ denote the two-dimensional norm. BT denote the boundary terms that will be discussed below. The dissipation $DI \leq 0$ and given by

$$DI = -2\nu\left(u^T(M_x H_y + H_x M_y)u + v^T(M_x H_y + H_x M_y)v\right).$$

The term $RT = u^T(A_x + B_y)\bar{H}u + v^T(A_x + B_y)\bar{H}v + (D_x u + D_y v)^T\bar{H}p$ is proportional to the discrete divergence. To obtain (41) we used the two assumptions: i) H is a diagonal norm, and ii) M_x, M_y are symmetric positive semi-definite are required. Note also that $(A_x + B_y)$ is equivalent to $(D_x u + D_y v)$ in the nonlinear case.

The boundary terms ($BT \equiv BT_w + BT_e + BT_s + BT_n$) are given by

$$\begin{array}{rllll}
BT_w = & +u_w^T H\left(u_w - 2\tau_w^{(u)}\lambda_5^{(w)}\right)u_w & +2u_w^T H\left(1+\tau_w^{(u)}\right)(p_w - \nu(Su)_w) \\
& +v_w^T H\left(v_w - 2\tau_w^{(v)}\lambda_2^{(w)}\right)v_w & +2v_w^T H\left(1+\tau_w^{(v)}\right)(-\nu(Sv)_w) \\
BT_e = & -u_e^T H\left(u_e - 2\tau_e^{(v)}\lambda_5^{(e)}\right)u_e & -2u_e^T H\left(1-\tau_e^{(u)}\right)(p_w - \nu(Su)_e) \\
& -v_e^T H\left(v_w - 2\tau_e^{(v)}\lambda_2^{(e)}\right)v_e & -2v_e^T H\left(1-\tau_e^{(v)}\right)(-\nu(Sv)_e) \\
BT_s = & +v_s^T H\left(v_s - 2\tau_s^{(v)}\lambda_5^{(s)}\right)v_s & +2v_s^T H\left(1+\tau_s^{(v)}\right)(p_s - \nu(Sv)_s) \\
& +u_s^T H\left(u_s - 2\tau_s^{(u)}\lambda_2^{(s)}\right)u_s & +2u_s^T H\left(1+\tau_s^{(u)}\right)(-\nu(Su)_s) \\
BT_n = & -v_n^T H\left(v_n - 2\tau_n^{(u)}\lambda_5^{(n)}\right)v_n & -2v_n^T H\left(1-\tau_n^{(v)}\right)(p_n - \nu(Sv)_n) \\
& -u_n^T H\left(u_n - 2\tau_n^{(u)}\lambda_2^{(n)}\right)u_n & -2u_n^T H\left(1-\tau_n^{(u)}\right)(-\nu(Su)_n)
\end{array}$$

BT can clearly be bounded if and only if $\tau_{w,s}^{(u,v)} = -1$ and $\tau_{e,n}^{(u,v)} = 1$. That leads to,

$$BT = \left\{\begin{array}{ll}
-u_w^T\left(H\sqrt{u_w^2+8}\right)u_w & -v_w^T\left(H\sqrt{v_w^2+4}\right)v_w \\
-u_e^T\left(H\sqrt{u_e^2+8}\right)u_e & -v_e^T\left(H\sqrt{v_e^2+4}\right)v_e \\
-v_s^T\left(H\sqrt{v_s^2+8}\right)v_s & -u_s^T\left(H\sqrt{u_s^2+4}\right)u_s \\
-v_n^T\left(H\sqrt{v_n^2+8}\right)v_v & -u_n^T\left(H\sqrt{u_n^2+4}\right)u_n\ ,
\end{array}\right. \tag{42}$$

and an energy estimate completely analog to the continuous case. (To realize that $||C^+||_\Gamma^2 = \sum_{i=1,3,4}\oint_\Gamma \lambda_i C_i^2 ds$ in (18) correspond to (42), solve for the gradients in (17) and insert them in $||C^+||_\Gamma^2$).

4.3 Solving the Poisson system for the pressure

To close the system (36) requires appropriate boundary condition for the pressure. We will very briefly outline the procedure to compute a modified pressure that is consistent with the maximally dissipative divergence boundary condition (37) and the momentum boundary conditions (38),(39). The procedure for the west (or the east) boundary goes like this

1. Use (39) to extract the tangential component of velocity ($\tilde{v}_w$) at the boundary.
2. Use (37) and $\tilde{v}_w$ to compute the normal component of velocity ($\tilde{u}_w$) at the boundary.
3. Use (38), $\tilde{v}_w$ and $\tilde{u}_w$ to compute the pressure ($\tilde{p}_w$) at the boundary.

Similarly, the procedure for the south (or the north) boundary is given by

1. Use (38) to extract the tangential component of velocity ($\tilde{u}_s$) at the boundary.
2. Use (37) and $\tilde{u}_s$ to compute the normal component of velocity ($\tilde{v}_s$) at the boundary.
3. Use (39), $\tilde{u}_s$ and $\tilde{v}_s$ to compute the pressure ($\tilde{p}_s$) at the boundary.

Once the pressure at the boundaries are determined, we can close the Poisson system. First we multiply (36) with $\bar{H} \equiv H_x H_y$, to obtain $((-M + BS \otimes H) + (H \otimes -M + BS))\, p = \bar{H} f$. Next we use $\tilde{p}_{w,\,e,\,s,\,n}$ (obtained in the procedure described above) to reduce it to a system where only the interior points are included (denoted by the tilde sign). The resulting non-singular positive definite symmetric reduced system can be written

$$\left(\tilde{M} \otimes \tilde{H} + \tilde{H} \otimes \tilde{M}\right) \tilde{p} = -\widetilde{\bar{H}\, f} - BT, \tag{43}$$

where $BT = (\tilde{M}_w \otimes \tilde{H})\tilde{p_w} + (\tilde{M}_e \otimes \tilde{H})\tilde{p_e} + (\tilde{H} \otimes \tilde{M}_s)\tilde{p_s} + (\tilde{H} \otimes \tilde{M}_n)\tilde{p_n}$. The matrices $\tilde{M}_{w,\,s}$ and $\tilde{M}_{e,\,n}$ denote the first and last column of M, excluding the first and last row.

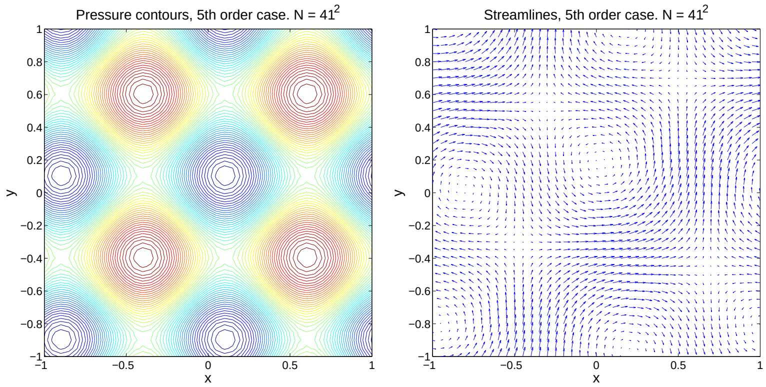
Remark: The nonzero elements in the boundary derivative operator BS (see Definition 4.2) sits in the first and the last row. The dissipative part M is by construction symmetric and positive semi-definite. By removing the first and last rows and the first and last columns in $-M + BS$, it is reduced to $-\tilde{M}$. It can be shown that $\tilde{M}$ is positive definite. Due to the symmetric positive definiteness of (43), it can be solved efficiently. We use the LU decomposition technique.

5 Computations

We will test our method on the analytic Taylor vortex model given by,

$$\begin{aligned}
u &= -cos(\alpha)sin(\beta)exp(-2\pi^2\nu t) + u_\infty \cos(\theta) \\
v &= sin(\alpha)cos(\beta)exp(-2\pi^2\nu t) + u_\infty \sin(\theta) \\
p &= -\tfrac{1}{4}\left(cos(2\,\alpha) + cos(2\,\beta)\right) exp(-4\pi^2\nu t) \\
\alpha &= \pi(x - x_0 - u_\infty \cos(\theta)t)\ ,\ \beta = \pi(y - y_0 - u_\infty \sin(\theta)t).
\end{aligned} \tag{44}$$

The Taylor vortex is an analytic solution to the incompressible Navier-Stokes equations. In (44), $(x_0,\, y_0)$ is the initial position, u_∞ is the free-stream value and θ the direction of the



(a) Pressure (b) Streamlines

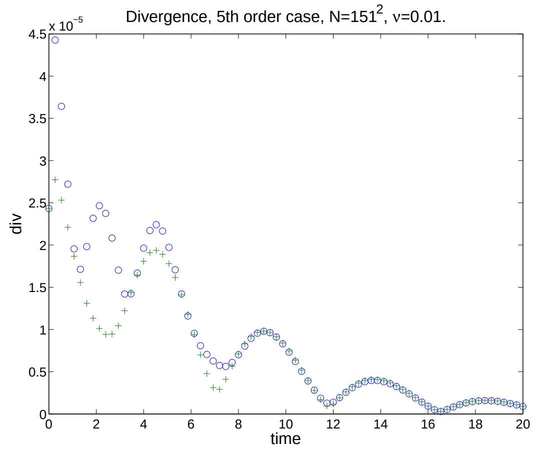
Figure 2: Pressure contour and streamlines at T=0.4, $N = 41^2$, $\nu = 0.01$

flow. The vortex is introduced into the computational domain by using the analytic solution as boundary data and initial data. For time advancement we use the standard 4th order Runge-Kutta method. In all the calculations below we use $\nu = 0.01$ and a 5th order scheme (6th order interior accuracy and 3rd order at boundaries). The pressure and streamline contours calculated on the 41^2 mesh at $T = 0.4$ are shown in Figure 2.

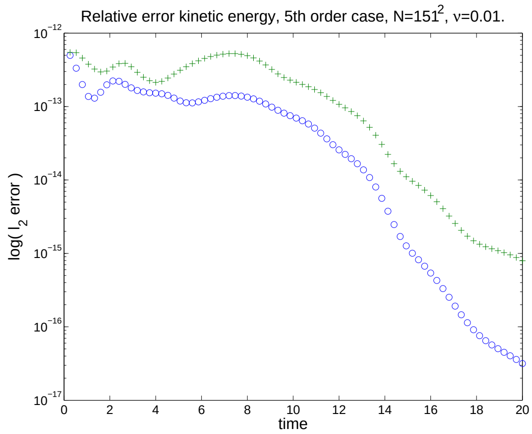
The convergence results at $T = 1$ for u, v, p and the divergence are shown in Tables 1 and 2. The divergence is calculated using $D_x u + D_y v$ with the 5th order scheme. Both calculations use (38) and (39) as boundary conditions. BC0 uses a Dirichlet condition on the divergence as the third condition. BC1 uses the advection-diffusion condition (37) as the third condition. Both conditions are acceptable in the sense that they do not contribute to growth of the divergence for the continuous problem.

N	$log(l_2(u))$	$q^{(u)}$	$log(l_2(v))$	$q^{(v)}$	$log(l_2(p))$	$q^{(p)}$	$log(l_2(div))$	$q^{(div)}$
21	-2.50	0.00	-2.49	0.00	-2.46	0.00	-1.30	0.00
41	-3.85	4.51	-3.92	4.76	-3.90	4.79	-2.55	4.17
61	-4.67	4.65	-4.75	4.73	-4.76	4.88	-3.27	4.07
81	-5.29	4.92	-5.37	4.93	-5.37	4.87	-3.75	3.87
101	-5.77	5.02	-5.85	5.02	-5.84	4.85	-4.11	3.71
151	-6.67	5.15	-6.74	5.10	-6.68	4.81	-4.74	3.61

Table 1: l_2-error and convergence rate q. Dirichlet boundary condition. $\nu = 0.01$



(a) Divergence



(b) Kinetic energy

Figure 3: Time history for divergence and l_2 error of kinetic energy, $N = 151^2$. Circles are BC1. $\nu = 0.01$.

N	$log(l_2(u))$	$q^{(u)}$	$log(l_2(v))$	$q^{(v)}$	$log(l_2(p))$	$q^{(p)}$	$log(l_2(div))$	$q^{(div)}$
21	-2.09	0.00	-2.09	0.00	-2.23	0.00	-0.62	0.00
41	-3.87	5.90	-3.86	5.89	-4.03	6.01	-2.15	5.09
61	-4.84	5.51	-4.83	5.51	-5.04	5.72	-3.03	4.98
81	-5.50	5.33	-5.50	5.35	-5.70	5.31	-3.61	4.59
101	-6.02	5.29	-6.01	5.31	-6.20	5.13	-4.02	4.26
151	-6.95	5.32	-6.95	5.38	-7.10	5.15	-4.71	3.95

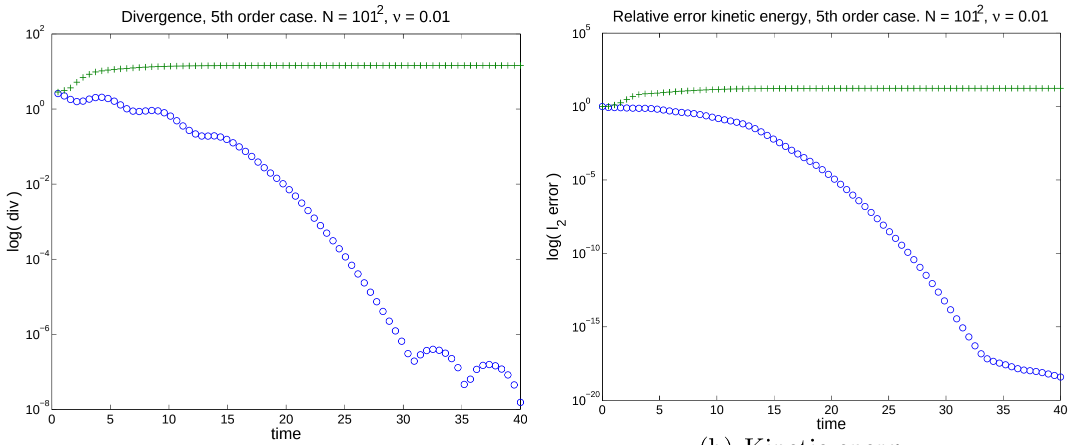
Table 2: l_2-error and convergence rate q. Advection-diffusion BC for divergence. $\nu = 0.01$

The correct order of accuracy is obtained for all variables. Also, as could be expected, the order of accuracy of the divergence seem to be approximately one order less compared to the one obtained for the variables. The results are similar with a slight advantage for BC1.

The time evolution (decay as predicted by the theory in this paper) of the divergence and l_2-error of the kinetic energy are shown in Figure 3. Both calculations were initiated using the exact analytical solution (44) injected in the grid points. The result is very similar for the two types of boundary conditions and in both cases, the divergence is clearly decreasing. The result of the two calculations at $T = 20$ is shown in Table 3. The results using BC1 is marginally better.

BC	$log(l_2(u))$	$log(l_2(v))$	$log(l_2(p))$	$log(l_2(div))$	$log(l_2(u^2 + v^2))$
$BC0$	-8.41	-8.23	-8.67	-6.42	-16.14
$BC1$	-9.23	-8.94	-9.51	-6.43	-17.54

Table 3: l_2-error at $t = 20$, $N = 151^2$, $\nu = 0.01$. Two different boundary conditions.



(a) Divergence (b) Kinetic energy

Figure 4: Time history for divergence and l_2 error of kinetic energy, $N = 101^2$. The initial solution is far from the exact solution. Circles are BC1. $\nu = 0.01$.

As a final test we initialized the two calculations above with the initial data $u = v = 0$ and $p = 1$. We used exact boundary data given by (44). This means that the initial data is far from the exact solution. The time evolution of the divergence and l_2-error of the kinetic energy are shown in Figure 4. This time the result using BC1 is clearly superior with a fast convergence towards the correct solution. BC0 does not converge at all. The result of the two calculations at $T = 40$ is shown in Table 4. The convergence of the calculation using BC1 is further illustrated in Figures 5, 6 and 7. This final test illustrates the remarks on page 4 and 5 which states that initial non-zero divergence effects might decay with time by using a sufficiently dissipative boundary condition.

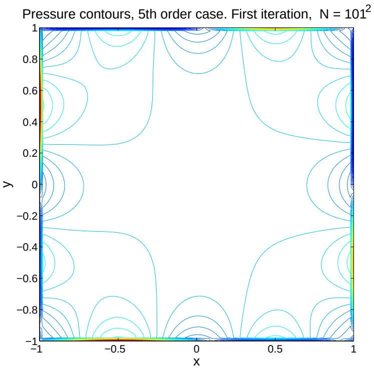
BC	$log(l_2(u))$	$log(l_2(v))$	$log(l_2(p))$	$log(l_2(div))$	$log(l_2(u^2+v^2))$
$BC0$	-0.09	-0.09	0.02	0.79	0.20
$BC1$	-9.97	-9.97	-10.81	-8.19	-19.47

Table 4: l_2-error at $t = 20$, $N = 151^2$, $\nu = 0.01$. Two different boundary conditions.

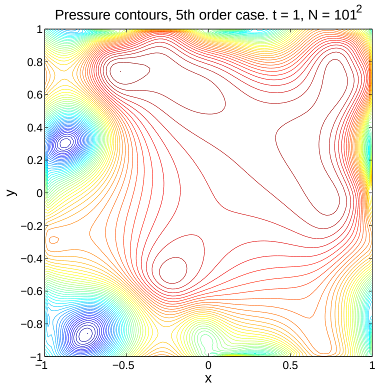
6 Conclusions

The nonlinear velocity-pressure formulation of the incompressible Navier-Stokes equations is analyzed. New sets of boundary conditions are derived. The boundary conditions have the same form on both inflow and outflow boundaries and lead to a divergence free solution.

The new boundary conditions are applied to a constant coefficient problem where the solution in Laplace-Fourier space can be written down explicitly. It is shown that the velocity-pressure formulation with the new boundary conditions yield a divergence free solution and

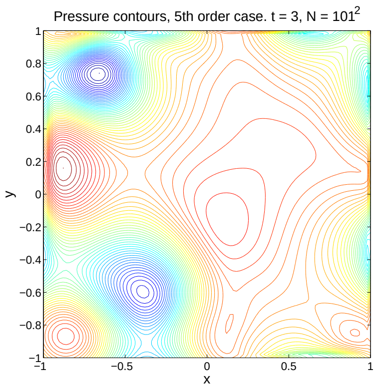


(a) Pressure, T=Δt

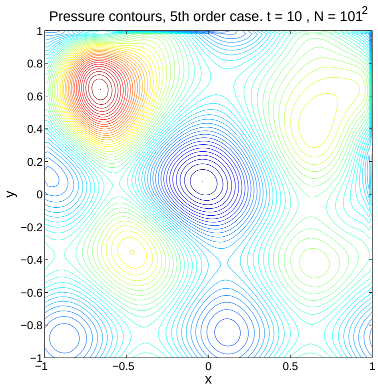


(b) Pressure, T=1

Figure 5: Snapshots of pressure distribution using BC1, $N = 101^2$, $\nu = 0.01$.

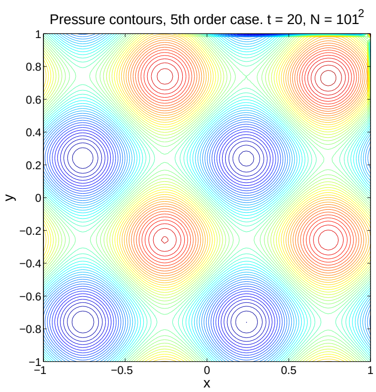


(a) Pressure, T=3

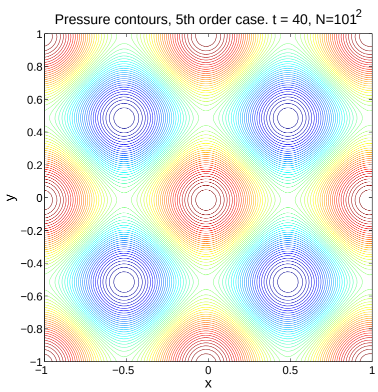


(b) Pressure, T=10

Figure 6: Snapshots of pressure distribution using BC1, $N = 101^2$, $\nu = 0.01$.



(a) Pressure, T=20



(b) Pressure, T=40

Figure 7: Snapshots of pressure distribution using BC1, $N = 101^2$, $\nu = 0.01$.

that all dependent variables including the pressure is bounded.

The new formulation including the new boundary conditions were implemented numerically using summation-by-parts operators and penalty technique. Stability of the linearized problem was shown. It was also shown how to use the new boundary conditions to obtain consistent pressure values on the boundary necessary for obtaining a symmetric positive definite Poisson system.

The numerical procedure was tested on an exact analytical solution to the Navier-Stokes equations and stability as well as the correct order of accuracy were demonstrated. Also, as predicted by the theory, the initial divergence in the solution decays, and as time passes a more and more divergence free and accurate solution is obtained. This process is much more pronounced for the most dissipative variant of the boundary conditions. In that case, convergence to the true divergence free solution was obtained although the initial condition was far from the exact solution.

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