New Input-Output Pairing Strategies Based on Linear Quadratic Gaussian Control

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Abstract

Two input-output pairing strategies based on linear quadratic Gaussian (LQG) control are suggested. In the first strategy, denoted linear quadratic interaction index (LQII), input-output pairing suggestions are found from a minimization of the output signal variance. This index not only guides to what pairing should be tried, it also gives a direct measure of how much better a full MIMO controller can perform. The second proposed interaction measure, denoted integrating linear quadratic index array (ILQIA), focuses more on the low frequency behaviour of the considered plant, such as load disturbances. The strategies are used to compare the expected performance of decentralized control structures in some illustrative examples. The pairing suggestions are compared with the recommendations previously obtained using other interaction measures such as the relative gain array (RGA), the Hankel interaction index array (HIIA) and the participation matrix (PM). The new strategies are easy to interpret and give suitable pairing recommendations where other methods may fail.

Keywords: decentralized control; linear quadratic regulators; LQG control; minimum variance control; multi-input/multi-output systems; multivariable control systems.

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1. Introduction

A common problem for systems with many input and output signals is channel interaction, *i.e.* when one input affects several outputs. Therefore, the expected level of coupling between the selected control loops is essential to establish prior to the control structure selection. Today there are several interaction measures that assist in this selection. One of the most commonly used is the relative gain array (RGA) introduced in [5]. Later, several extensions to and modifications of the RGA have been introduced, see for instance [24] for a review. Other, more recent interaction measures include the decentralized relative gain (dRG) [30], the effective relative gain array (ERGA) [36] and the Gramian-based measures; the Hankel interaction index array (HIIA) [34] and the participation matrix (PM) [6, 28]. Further, an $\mathcal{H}_2$-norm based interaction measure denoted Σ_2 was introduced in [4] and further analysed in [12]. One advantage of these measures is that they are not restricted to decentralized control structures, which is one of the main restrictions of the RGA.

In control performance assessment, the lowest possible output variance, the minimum variance, has been a key component in many benchmark studies since the work of Harris [15] more than two decades ago. The main idea is to compare the calculated minimum variance with the actual achieved output variance and thereby get an indication of the current performance of the controller. Some extensions to the original idea in [15] are made in [16, 19, 21, 35, 33] and a review of the research that has followed in this field is given in [17]. Performance indices used for control structure selection and performance assessment for disturbance rejection in multiple-input multiple-output (MIMO) processes are proposed in [22].

Inspired by the above mentioned work in the field of control performance assessment, two pairing strategies based on linear quadratic Gaussian (LQG) control are proposed. The key idea for the first measure, denoted linear quadratic interaction index (LQII), is to design single-input single-output (SISO) LQG controllers for each input-output pairing and thereafter form the closed-loop MIMO system. The closed-loop performance in terms of the output variance is then computed for each control structure and the pairing corresponding to the lowest output variance is selected. The proposed index also gives a direct measure of how much better a full MIMO controller can perform. As an alternative, minimum variance controllers can be used (see [13]). In the second measure, denoted Integrating Linear Quadratic Index Array (ILQIA), the integral feedback gain in an LQG design with integral action is considered. This way the focus is on the low frequency behaviour of the plant, such as load disturbance rejection.

The structure of this paper is as follows: The two-by-two system that is considered in the theoretical derivations is described in Section 2. In Section 3 the controllers are designed and the expressions for the full closed-loop MIMO systems for decentralized control are derived. The proposed pairing strategies are presented in Section 4. In Section 5 different interaction measures are compared in some illustrative examples. Finally, the conclusions are drawn in Section 6.

2. System description

The methods presented in the following apply to square systems of arbitrary size. For illustration, consider a stable linear discrete-time MIMO system with two inputs, u_1 and u_2, and two outputs, y_1 and y_2. The transfer function matrix $G(\mathrm{q})$ of the system can be partitioned as

$$G(\mathrm{q}) = \begin{bmatrix} G_{11}(\mathrm{q}) & G_{12}(\mathrm{q}) \\ G_{21}(\mathrm{q}) & G_{22}(\mathrm{q}) \end{bmatrix}. \tag{1}$$

Figure 1 gives a graphical representation of the considered system (noise excluded) and its cross-couplings. The subsystems $G_{12}(\mathrm{q})$ and $G_{21}(\mathrm{q})$ represent the cross-couplings if y_1 is controlled by u_1 and y_2 by u_2. The subsystems $G_{ij}(\mathrm{q})$ can be equivalently expressed in state space form represented by the system matrices $(A_{ij}, B_{ij}, C_{ij}, 0)$. In the design of the controllers, it is assumed that process noise and measurement noise are present.

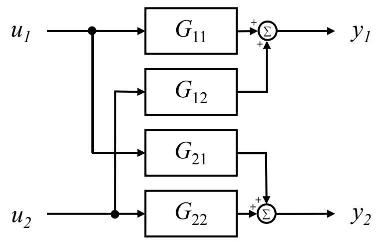

Figure 1: Block diagram of the considered system.

A state-space representation of the full system is thus given by

$$\begin{aligned} x(t+1) &= A_f x(t) + B_f u(t) + N_f v(t), \\ y(t) &= C_f x(t) + e(t). \end{aligned} \tag{2}$$

where

$$\begin{aligned} x(t) &= \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix}, A_f = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & A_{12} & 0 & 0 \\ 0 & 0 & A_{21} & 0 \\ 0 & 0 & 0 & A_{22} \end{bmatrix}, \\ B_f &= \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \\ B_{21} & 0 \\ 0 & B_{22} \end{bmatrix}, C_f = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 \\ 0 & 0 & C_{21} & C_{22} \end{bmatrix}, \\ y(t) &= [y_1(t) \;\; y_2(t)]^T. \end{aligned}$$

$\{v(t)\}$ and $\{e(t)\}$ are white noise sequences with diagonal covariance matrices, *i.e.* the noise components are assumed to be uncorrelated. The intensities are R_v and R_e, respectively. Observe that all matrices may be block matrices and consequently 0 may denote matrices with zeros in all elements.

3. Control design

For the determination of the LQII, an LQG controller is designed for each SISO subsystem G_{ij}. The n^2 subsystems are then connected to form $n!$ different MIMO closed-loop systems. Different control structures, *i.e.* different pairings of the inputs and outputs, therefore give different closed-loop expressions for the full system, which are derived in this section. The LQG design procedure guarantees stable closed-loop systems for each SISO subsystem. There is, however, no guarantee that the full closed-loop system will be stable and, therefore, the stability has to be investigated separately.

3.1. Linear quadratic Gaussian (LQG) control

Optimal linear control for a general SISO system $(A, B, C, 0)$ can be obtained by minimizing the criterion [9, 25]

$$V = \mathrm{E} \sum_{t} \big(x^T(t) Q_x x(t) + u^T(t) Q_u u(t) \big), \tag{3}$$

where E is the expectation operator. In the following, the first weight in the criterion (3) is chosen as $Q_x = C^T Q_y C$ in order to penalize the outputs rather than the states. This gives the criterion

$$\tilde{V} = \mathrm{E} \sum_{t} \big(y^T(t) Q_y y(t) + u^T(t) Q_u u(t) \big). \tag{4}$$

The following LQG control law is used:

$$\begin{aligned} u(t) &= -L\hat{x}(t|t) \\ &= -L(I - MC)\hat{x}(t) - LMy(t), \qquad (5) \\ \hat{x}(t+1) &= A\hat{x}(t) + Bu(t) + K\big(y(t) - C\hat{x}(t)\big), \qquad (6) \end{aligned}$$

where $\hat{x}(t)$ is the estimate of $x(t)$ using measurements up to time instant $t-1$, $\hat{x}(t|t)$ is the corresponding estimate that uses measurements up to time instant t, M is the innovation update gain obtained as

$$M = PC^T(CPC^T + R_e)^{-1}, \tag{7}$$

and K is the Kalman gain given by

$$K = APC^T(CPC^T + R_e)^{-1}, \tag{8}$$

where P is the positive semidefinite solution to the discrete-time Riccati equation

$$P = APA^T + NR_vN^T - APC^T(CPC^T + R_e)^{-1}CPA^T. \tag{9}$$

L is the optimal gain given by

$$L = (B^T SB + Q_u)^{-1} B^T SA, \tag{10}$$

where S is the positive semidefinite symmetric solution to the discrete-time Riccati equation

$$S = A^T SA + Q_x - A^T SB(B^T SB + Q_u)^{-1} B^T SA. \tag{11}$$

To guarantee the existence of unique solutions to the Ricatti equations (10) and (11), it is assumed that (A, B) is stabilizable and that (A, Q_x) is detectable. However, if the state space model originates from a transfer function, a minimal realization automatically fulfills these conditions. Each SISO closed-loop system can be expressed as

$$\begin{bmatrix} x(t+1) \\ \hat{x}(t+1) \end{bmatrix} = F \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + J \begin{bmatrix} v(t) \\ e(t) \end{bmatrix},$$

$$y(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} v(t) \\ e(t) \end{bmatrix}, \tag{12}$$

where the LQG controller gives

$$\begin{aligned} F &= \begin{bmatrix} A - BLMC & -BL(I - MC) \\ KC - BLMC & A - KC - BL(I - MC) \end{bmatrix}, \\ J &= \begin{bmatrix} N & -BLM \\ 0 & K - BLM \end{bmatrix}. \end{aligned} \tag{13}$$

For a derivation of the full MIMO closed-loop systems for the two decentralized pairing options y_1–u_1, y_2–u_2, and y_1–u_2, y_2–u_1, respectively, of a 2×2 system, see [14].

3.2. Integral action

Integral action for setpoint tracking can be incorporated in state space design in many different ways (see for example [2, 9, 25]). One straightforward way is to extend the state space model with integral states

$$
\begin{aligned}
x_I(t) &= \frac{1}{\mathrm{q}-1}(r(t)-y(t)) \\
&= \frac{1}{\mathrm{q}-1}(r(t)-Cx(t)-e(t))
\end{aligned} \tag{14}
$$

such that

$$
\begin{aligned}
x_e(t+1) &= A_e x_e(t) + B_e u(t) + Hr(t) + N_e \begin{bmatrix} v(t) \\ e(t) \end{bmatrix}, \\
y(t) &= C_e x_e(t) + e(t),
\end{aligned}
$$

where $x_e(t) = \begin{bmatrix} x(t) & x_I(t) \end{bmatrix}^T$,

$$
\begin{aligned}
A_e &= \begin{bmatrix} A & 0 \\ -C & I \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \\
H &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C_e = \begin{bmatrix} C & 0 \end{bmatrix}, \quad N_e = \begin{bmatrix} N_f & 0 \\ 0 & -I \end{bmatrix},
\end{aligned}
$$

and $r(t)$ is the reference signal. The optimization criterion (3) is then extended to include the integral states:

$$
V = \mathrm{E} \sum_t \left(x_e^T(t) \begin{bmatrix} Q_x & 0 \\ 0 & Q_I \end{bmatrix} x_e(t) + u^T(t) Q_u u(t) \right), \tag{15}
$$

where $x_e(t)$ is the extended state vector. Solving the corresponding Riccati equations (10) and (11) will then give the state feedback

$$
\begin{aligned}
u(t) &= -Lx_e(t) \\
&= -L_x x(t) - L_I x_I(t).
\end{aligned} \tag{16}
$$

The larger Q_I is compared to Q_x and Q_u, the more integral action through the gain L_I can be expected.

For continuous-time systems the following integral states are introduced instead:

$$
x_I(t) = \frac{1}{\mathrm{p}}(r(t)-y(t)) = \frac{1}{\mathrm{p}}(r(t)-Cx(t)-e(t)). \tag{17}
$$

where p is the differentiation operator. Compared to the discrete-time counterpart, the only state-space matrix that differs is the extended A-matrix, which for the continuous-time case is

$$A_e = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}. \tag{18}$$

The continuous-time optimization criterion is

$$V = \mathrm{E} \int_t \big(x^T(t) Q_x x(t) + u^T(t) Q_u u(t) \big) \mathrm{d}t. \tag{19}$$

The corresponding criterion for the extended model is

$$V = \mathrm{E} \int_t \big(x_e^T(t) \begin{bmatrix} Q_x & 0 \\ 0 & Q_I \end{bmatrix} x_e(t) + u^T(t) Q_u u(t) \big) \mathrm{d}t. \tag{20}$$

4. Control structure selection

4.1. Scaling and variances

As a first step, before the calculations leading to the control structure selection, the model inputs and outputs must be properly scaled. The methods that will be introduced are based on comparisons of the sum of the individual output variances resulting from a sum of input variances. Therefore, the outputs should be scaled such that the importance of unit variance of each scaled output should be equal. In other words, if σ_i^2 are set to be the levels of acceptable individual variances the scaled outputs should be $y_i = \tilde{y}_i / \sigma_i$, where $\tilde{y}_i$ are the original outputs. The inputs are first scaled in the same way, and then a common scaling can be applied to all the inputs such that the cost of the input variances and the cost of the output variances are reasonably balanced.

Next, for the determination of LQII the variances for the disturbances must be set. If the original system is already in a state space form and their intensities are known, then those should naturally be used. Otherwise, the default is to set $N_f = B$ and $R_v = R_e = I$.

The settings for the cost matrices in (4), (15) and (19) are discussed in context with the two proposed indices.

4.2. Linear quadratic interaction index (LQII)

As a measure of the performance of the considered control structure, the sum of the output variances for the closed-loop MIMO system is used here. This measure gives an indication of how appropriate the control structure is compared to other structures. The control structure that gives the smallest sum is the structure that has the most desirable pairing combination in a minimum variance sense, and is therefore the suggested input-output pairing. Note again, that the pairing strategy is not limited to decentralized control structures though only decentralized structures are compared in this paper.

The output variances can be calculated in the following way: Write the full MIMO closed-loop system as

$$\begin{aligned} X(t+1) &= F_{CL}X(t) + J_{CL}V(t), \\ y(t) &= C_{CL}X(t) + T_{CL}V(t), \end{aligned} \tag{21}$$

where the noise vector $V(t)$ has the covariance R_V. The stationary state covariance matrix $\Pi = \mathrm{E}X(t)X^T(t)$ can then be determined by solving the discrete-time Lyapunov equation [32]

$$\Pi = F_{CL}\Pi F_{CL}^T + J_{CL}R_V J_{CL}^T. \tag{22}$$

The stationary output covariance matrix for this system is then given by

$$\mathrm{E}y(t)y^T(t) = C_{CL}\Pi C_{CL}^T + T_{CL}R_V T_{CL}^T. \tag{23}$$

To compare the expected control performance of a specific control structure with the full MIMO controller, the following ratio, denoted linear quadratic interaction index (LQII), will be used:

$$\text{LQII} = \frac{\sum_i var\{y_i\}}{\sum_i var\{y_i^{\text{MIMO}}\}} \geq 1. \tag{24}$$

Since the full MIMO controller gives the lowest possible sum of the output variances (the denominator), the LQII gives an indication of how much worse the performance is expected to be for a specific control structure compared to the full MIMO structure in terms of output variance. Ideally, the selected control structure should be as simple as possible, but still give satisfactory performance. If the performance criterion is a minimization of the output variances, this means that the LQII should be as close to 1 as possible.

Due to the scaling, the cost matrices in (4) are $Q_y = I$ and $Q_u = q_u I$. Setting q_u to a very small value results in an LQG control very close to minimum variance control, but with the advantage of not being restricted to minimum-phase systems. The index LQII then tells us what are the lowest possible output variances we can get using SISO controllers compared to using a full MIMO controller. Setting $q_u = 1$ results in a more realistic LQG controller that has the potential of giving a feedback resulting in variances slightly higher than achieved by minimum variance control, but with substantially lower variances of the control signals [2]. This is also confirmed in the examples that follow.

4.3. Integrating linear quadratic index array (ILQIA)

The method suggested above is exhaustive in the sense that it suggests the pairing that gives the smallest variance out of all possible pairings. As a consequence the computational burden grows as $n!$, which for large systems is disadvantageous compared to RGA or the Gramian based interaction measures. Further, load disturbance rejection and servo properties may be overlooked since it is only variances that are considered.

Assume the control system is primarily designed to minimize the effects of low frequency disturbances and consider a SISO control system with, for example, a PID controller

$$F(s) = K_P + K_I s^{-1} + K_D s$$

and disturbances v_1 and v_2 (see Figure 2). An appropriate measure of the low frequency (LF) performance is then an LF weighted response from v_1 to y, *e.g.* $||s^{-1}S(s)G(s)||_\infty$, where S is the sensitivity function (see [8] and references therein). Due to the integral action, the sensitivity function behaves as $s/(K_I G(s))$ for low frequencies and hence, minimizing the measure corresponds approximately to minimizing $1/K_I$.

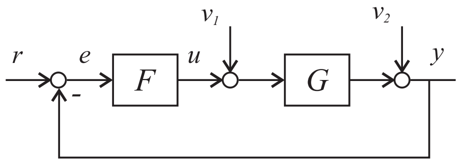

Figure 2: SISO feedback control system.

In the time-domain, applying the final value theorem to the control error after a step disturbance v_1 gives

$$\lim_{t\to\infty}\int_0^t e(\tau)d\tau = \lim_{s\to 0}\frac{-G(s)}{1+F(s)G(s)}\frac{1}{s} = -\frac{1}{K_I},$$

which equals (except for the sign) the previous LF measure. Similarly, for a step disturbance in v_2

$$\lim_{t\to\infty}\int_0^t e(\tau)d\tau = \lim_{s\to 0}\frac{-1}{1+F(s)G(s)}\frac{1}{s} = -\frac{1}{K_I G(0)}.$$

provided $G(0) \neq 0$. Hence, the higher the integral gain in the feedback is the better the load disturbance rejection generally is (as also noted by Åström and Hägglund [1]). In fact, the strong dependence of the low frequency performance on the integral gain holds quite independently of model uncertainties and other control specifications [8].

With focus on disturbance rejection and setpoint tracking, a different approach, still based on LQ design, is applied by considering the integral action feedback L_I in (16). By setting $Q_x = 0$ and $Q_I = I$ the elements of L_I are the optimal integral gains between outputs and inputs for a control activity determined by q_u. As stated, the larger the integral gain that can be applied in each SISO loop, the better the LF performance can be expected. As a consequence the input-output pairing that gives the largest individual integral gains is sought.

For this purpose the Integrating Linear Quadratic Index Array (ILQIA) is defined as the normalized optimal integral gains, *i.e.* the element (i,j) in the matrix $ILQIA$ is

$$[ILQIA]_{ij} = \frac{|[L_I]_{ij}|}{\sum_{k,l}|[L_I]_{kl}|}. \tag{25}$$

The sum of all the elements in $ILQIA$ is 1, and the aim is to find the selection of (i, j) that gives a sum as close to 1 as possible.

4.4. Stability considerations

One way of detecting possible instability issues of a specific pairing selection is to adopt two of the pairing rules from the RGA analysis [20, 9]. The first rule is to avoid pairings that correspond to negative RGA elements. Secondly, decentralized pairings that give a negative Niederlinski Index (NI) should be avoided. NI is defined as

$$\mathrm{NI}(G) = \det\overline{G}(0) \Big/ \prod_{i=1}^{n} \overline{g}_{ii}(0), \tag{26}$$

where $\overline{G} = [\overline{g}_{ij}]$ is the transfer function matrix of the considered system where the inputs and outputs have been re-arranged such that the considered decentralized pairing is found along the diagonal. A negative NI together with the assumptions [24] that (i) all subsystems $\overline{g}_{ij}$ are rational and proper, (ii) each feedback control loop contains an integrating action and (iii) each individual control loop remains stable when any of the other loops are opened, indicates that the closed loop system is unstable. For a detailed description of the NI and RGA pairing procedure, see for instance [10, 24, 31]. These rules can be incorporated in the pairing procedures based on the two proposed pairing strategies, as well as in procedures based on measures such as the HIIA, the PM and the Σ_2. He and Cai [18] and Fatehi and Shariati [7], for example, include the NI in their ERGA and Normalized RGA pairing algorithms.

5. Examples

In this section simulation examples taken from the literature on input-output pairing are presented, where the proposed LQG control pairing strategies are used in order to decide appropriate decentralized controller structures. The resulting suggestions are then compared with the ones obtained using other interaction measures, such as the RGA, the HIIA, the PM and Σ_2.

For the LQII calculation, all considered systems were sampled. Unless otherwise stated, the sampling period was 1 s. All the other interaction measures (including

ILQIA) were calculated for the continuous-time system. Third order (which was found to be sufficient) Padé-approximations are used for the time delays, and it is further assumed that all models have been properly scaled, as described in Section 4.1. Two input weight q_u settings are used: $q_u = 10^{-9}$, which in practise corresponds to an unlimited control activity and hence minimum output variances, and $q_u = 1$ that because of the scaling corresponds to a realistic LQG controller in terms of input activity. As it turns out, the pairings suggested by LQII and ILQIA were found to be the same for both q_u.

5.1. Example 1

In the first example the interactions present in a quadruple-tank system will be examined (see [23] for a general description of this process). The considered continuous-time linear minimum-phase model is given by the following state space matrices:

$$
\begin{aligned}
A &= \begin{bmatrix} -0.0159 & 0 & 0.159 & 0 \\ 0 & -0.0159 & 0 & 0.02651 \\ 0 & 0 & -0.159 & 0 \\ 0 & 0 & 0 & -0.02651 \end{bmatrix}, \\
B &= \begin{bmatrix} 0.05459 & 0 \\ 0 & 0.07279 \\ 0 & 0.0182 \\ 0.03639 & 0 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

The obtained (theoretical) output variances and values of LQII are given in Table 1. Clearly, the pairing combination y_1–u_1, y_2–u_2 results in the smallest output variances for both LQG control settings. Hence, in a minimum variance sense, this is the recommended pairing selection. Also note that the LQII is very close to 1 for the recommended pairing which means that the output variances of the suggested decentralized control structure are only slightly larger than for a full MIMO control structure. For this reason, decentralized control can be expected to work well for this plant. Moreover, the RGA elements and the NI are positive for y_1–u_1, y_2–u_2 but negative for the other decentralized pairing combination. Hence, no instability issues are indicated for the recommended control structure.

Table 1: LQII and output variances for LQG control in Example 1. The pairing is specified as ij where i is the output index and j the input index.

Pairing	q_u	$\sum_{i=1}^{2} var\{y_i\}$	LQII
11, 22	1	2.1729	1.0165
	$1 \cdot 10^{-9}$	2.1376	1.0171
12, 21	1	2.2891	1.0709
	$1 \cdot 10^{-9}$	2.2807	1.0852
Full MIMO	1	2.1376	1
	$1 \cdot 10^{-9}$	2.1016	1

For $q_u = 1$

$$ILQIA = \begin{bmatrix} 0.4812 & 0.0188 \\ 0.0188 & 0.4812 \end{bmatrix},$$

which clearly recommends the diagonal pairing. The sum of the diagonal elements of $ILQIA$ is 0.9624, and setting $q_u = 10^{-9}$ strengthen this recommendation even further with a sum of 0.9996. In [11] the RGA, the HIIA, the PM and the Σ_2 were used in the study of this process. All of these interaction measures recommend the same pairing for decentralized control.

5.2. Example 2

Now consider the continuous-time 2×2 process given by

$$G(s) = \begin{bmatrix} \frac{5e^{-40s}}{100s+1} & \frac{e^{-4s}}{10s+1} \\ \frac{-5e^{-4s}}{10s+1} & \frac{5e^{-40s}}{100s+1} \end{bmatrix},$$

where s is the Laplace variable. This process has been extensively analyzed, also by simulations using optimal decentralized PI controllers, with the conclusion that the off-diagonal pairing combination $y_1 - u_2$, $y_2 - u_1$ is preferred for decentralized control [26, 36]. The main reason for this is that the off-diagonal pairing combination corresponds to faster elements in G. The same pairing recommendation has also been concluded using the effective relative gain array (ERGA) [36], the Σ_2 measure [11, 12], and RGA evaluated at frequencies higher than 10^{-7} rad/s [26, 36]. However, static RGA, HIIA and PM all fail to conclude the best pairing.

The output variances and the LQII for both LQG control settings also indicate that the off-diagonal pairing combination is the most suitable since this pairing gives the lowest output variances (see Table 2). The ILQIA clearly recommends the off-diagonal pairing with

$$ILQIA = \begin{bmatrix} 0.0277 & 0.4723 \\ 0.4723 & 0.0277 \end{bmatrix}.$$

for $q_u = 1$ and a large off-diagonal pairing sum (0.84) also for "unbounded input" ($q_u = 10^{-9}$). All of the RGA elements and the NI for both of the decentralized pairings are positive.

Table 2: LQII and output variances for LQG control in Example 2. The pairing is specified as ij where i is the output index and j the input index.

Pairing	q_u	$\sum_{i=1}^{2} var\{y_i\}$	LQII
11, 22	1	3.4882	1.1408
	$1 \cdot 10^{-9}$	3.4742	1.1598
12, 21	1	3.2536	1.0641
	$1 \cdot 10^{-9}$	3.2084	1.0711
Full MIMO	1	3.0577	1
	$1 \cdot 10^{-9}$	2.9955	1

5.3. *Example 3*

The system considered is

$$G(s) = \frac{1}{1+s} \begin{bmatrix} e^{-s} & 1 \\ -1 & e^{-2s} \end{bmatrix},$$

where the sampling time was set to be 0.05 s. In this example $|G_{ij}(s)|$ is the same for all subsystems. Only the time delay and the sign of the gain differ, which makes it particularly hard for some of the considered interaction measures.

For the RGA and the Σ_2 all of the elements are equal. For the RGA this means that no conclusion can be drawn, and for the Σ_2 that all subsystems are equally important. As shown in [12], the Σ_2 does not react on time delays, and since this interaction measure can be interpreted as the area of the Bode magnitude diagram the resulting Σ_2 is not surprising. The HIIA and the PM are affected by the time delays in the diagonal elements, and recommend the diagonal pairing y_1–u_1 and y_2–u_2 as the decentralized paring. However, [27] found in simulation studies that the off-diagonal pairing should be the preferred decentralized choice. Balestrino, Crisostomi, Landi and Menicagli [3] use their suggested interaction measures ARGA and RoMA index to find this pairing choice. As seen in Table 3 the LQII agrees on this choice even though the difference in variance between the pairings is small. ILQIA is calculated to

$$ILQIA = \begin{bmatrix} 0.1393 & 0.3607 \\ 0.3607 & 0.1393 \end{bmatrix}$$

and, hence, also recommends the off-diagonal pairing with a sum of the off-diagonal elements of $ILQIA$ of 0.7215. However, removing the cost of control ($q_u = 10^{-9}$) reduces the sum to 0.52, indicating that a MIMO controller should be used then. The NI does not indicate any instability issues since it is positive for both pairings.

Table 3: LQII and output variances for LQG control in Example 3. The pairing is specified as ij where i is the output index and j the input index.

Pairing	q_u	$\sum_{i=1}^{2} var\{y_i\}$	LQII
11, 22	1	3.9885	1.0668
	$1 \cdot 10^{-9}$	3.9796	1.1783
12, 21	1	3.9422	1.0545
	$1 \cdot 10^{-9}$	3.8293	1.1338
Full MIMO	1	3.7386	1
	$1 \cdot 10^{-9}$	3.3775	1

5.4. Example 4

A continuous-time model of a bioreactor in a wastewater treatment plant is given by a state space description (A, B, C, D), where

$$A = [A_1|A_2|A_3],$$

$$A_1 = \begin{bmatrix} -26.72 & -99.89 & -0.1127 \\ 13.36 & -130.8 & 0 \\ 0.1002 & 0 & -26.97 \\ 0 & 115.2 & 13.36 \\ 0 & 0 & -2.102 \\ 0 & 0 & 0 \\ -0.4821 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -57.56 \\ -3.440 \cdot 10^{-5} & 0 \\ 17.50 & -117.1 \\ -13.36 & 0 \\ 0 & -1101 \\ -0.0006417 & 13.36 \\ 0 & -19.38 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -28.10 \\ -78.31 & 0 \\ 0 & 844.8 \\ -12.12 & 0 \\ 17.50 & -524.3 \\ -1474 & 0 \\ 0 & -3896 \end{bmatrix},$$

$$B = \begin{bmatrix} -200.1 & 0 \\ 100.1 & 0 \\ 195.7 & 0 \\ -97.88 & 0 \\ -9.660 & 1476 \\ 4.831 & 0 \\ 59.74 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0.3333 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3333 & 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This process has been analysed in [29] using the RGA and the HIIA, and the different pairings were also compared in control simulation experiments. The recommended decentralized pairing is y_1–u_1, y_2–u_2, which is also justified by process knowledge. The NI is positive for this pairing but negative for the off-diagonal pairing. The sampling period is selected to $1/1440$ day^{-1} and the obtained LQII values (see Table 4) when $q_u = 1$ are 1.8003 and 2.0147 for the diagonal and off-diagonal pairings, respectively. When $q_u = 1 \cdot 10^{-9}$ the corresponding values of LQII are 2.1783 and 2.6803. Clearly, the LQII values are relatively large for both of the decentralized pairing options, indicating that a full MIMO LQ controller is significantly better in terms of variance.

For $q_u = 1$

$$ILQIA = \begin{bmatrix} 0.3321 & 0.1679 \\ 0.1679 & 0.3321 \end{bmatrix}, \tag{27}$$

which hardly changes when $q_u = 10^{-9}$. Hence ILQIA also recommends diagonal pairing. However, all elements are of the same order of magnitude, indicating that a full multivariable control structure may be preferable. The bad performance of the decentralized controller was verified in the control simulations in [29], where the considered multivariable sparse controller suffered much less from cross-coupling than the

Table 4: LQII and output variances for LQG control in Example 4. The pairing is specified as ij where i is the output index and j the input index.

Pairing	q_u	$\sum_{i=1}^{2} var\{y_i\}$	LQII
11, 22	1	207.7027	1.8003
	$1 \cdot 10^{-9}$	179.1922	2.1783
12, 21	1	232.4377	2.0147
	$1 \cdot 10^{-9}$	220.4953	2.6803
Full MIMO	1	115.3732	1
	$1 \cdot 10^{-9}$	82.2639	1

decentralized one.

5.5. Example 5

Consider the 3×3 non-minimum phase process given by

$$G(s) = \frac{1-s}{(1+5s)^2} \begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix}.$$

This process is used in [20] as an example of when the static RGA does not recommend the most desirable pairing. The RGA recommends the diagonal pairing combination y_1–u_1, y_2–u_2 and y_3–u_3. However, as found in [20] this pairing combination is not suitable due to instability issues. Instead, they recommend the pairing combination y_1–u_2, y_2–u_3 and y_3–u_1. The same pairing suggestion was found in [18] when considering loop-by-loop interaction energy. The HIIA, the PM and the Σ_2 all recommend the pairing combination y_1–u_3, y_2–u_1 and y_3–u_2 for decentralized control. However, the RGA also indicates (by negative elements) that this pairing combination should be avoided (see for instance [20]). Moreover, the NI is negative for this particular pairing but positive for all of the others (see Table 5). If the HIIA, the PM and the Σ_2 are combined with the RGA rule of avoiding pairings corresponding to negative RGA elements (this is one component of the pairing rule used in [18]), the HIIA, the PM and the Σ_2 suggest the same pairing combination as the one recommended in [20] and [18].

Table 5: The sum of the elements of the HIIA, the Σ_2 and of the PM, respectively, the sum of output variances, LQII, the sum of the elements of the ILQIA, negative elements of the RGA and NI for different decentralized pairings for the plant in Example 5. The pairing is specified as ij where i is the output index and j the input index. The closer to 1 the sums of the considered interaction measures are, the better. In the calculation of LQII $q_u = 1 \cdot 10^{-9}$. The sum of the variances for the full MIMO controller is 20.5185.

Pairing:	11, 22, 33	11, 23, 32	12, 21, 33
HIIA and Σ_2	0.0446	0.4155	0.1691
PM	0.0021	0.4798	0.0404
$\sum_{i=1}^{3} var\{y_i\}$	75.7704	49.8111	74.2068
LQII	3.6928	2.4276	3.6166
ILQIA	0.2009	0.2925	0.2692
Neg. RGA-elem.	-	32	21
NI	26.94	1.04	1.04

Pairing:	12, 23, 31	13, 21, 32	13, 22, 31
HIIA and Σ_2	0.4629	0.4926	0.4155
PM	0.4916	0.5063	0.4798
$\sum_{i=1}^{3} var\{y_i\}$	49.3745	48.6842	49.8111
LQII	2.4063	2.3727	2.4276
ILQIA	0.3396	0.4594	0.4383
Neg. RGA-elem.	-	13, 21, 32	13
NI	0.25	-0.17	1.04

In the calculation of the LQII all six decentralized pairing combinations were evaluated. In Table 5 the sum of the output variances is presented for each combination. The LQII is minimized for the pairing combination y_1–u_3, y_2–u_1 and y_3–u_2 for both of the settings. Hence, it supports the recommendation made by the HIIA, the PM and the Σ_2. However, the pairing combination y_1–u_2, y_2–u_3 and y_3–u_1 also gives variances that are very close to the variances of the suggested pairing. If the pairing combinations corresponding to negative RGA elements and negative NI are avoided, the LQII gives the same pairing suggestion as the one recommended in [20, 18]. It was further found that the diagonal pairing recommended by the RGA is not desirable in a minimum variance sense. In fact, this pairing results in the largest sum of output variances (see Table 5). There are also two other pairings that give low output variances: pairing combination y_1–u_1, y_2–u_3 and y_3–u_2 and pairing combination y_1–u_3, y_2–u_2 and y_3–u_1. However, both of these invoke pairings corresponding to negative RGA elements. Note that all of the output variance sums for the decentralized pairings are much larger than the corresponding sum for the full MIMO control structure, indicating that it may be rewarding in terms of variance to seek a sparse, or full, MIMO control structure.

The ILQIA combined with the rule of avoiding negative RGA elements and negative NI gives the "correct" pairing y_1–u_2, y_2–u_3 and y_3–u_1. Otherwise, the ILQIA recommends the same pairing combination as the HIIA, the PM and the Σ_2. As for LQII, the relatively low value of the sum of integration gains and the small differences between the best pairings, ILQIA also indicates that MIMO control may improve performance significantly.

5.6. Example 6

Now consider the following distillation column model with four inputs and four outputs with transfer function matrix $G(s) = [G_1(s)\ \ G_2(s)]$ where

$$G_1(s) = \begin{bmatrix} \frac{-9.811e^{-1.59s}}{11.36s+1} & \frac{0.374e^{-7.75s}}{22.22s+1} \\ \frac{5.984e^{-2.24s}}{14.29s+1} & \frac{-1.986e^{-0.71s}}{66.67s+1} \\ \frac{2.38e^{-0.42s}}{(1.43s+1)^2} & \frac{0.0204e^{-0.59s}}{(7.14s+1)^2} \\ \frac{-11.3e^{-3.79s}}{(21.74s+1)^2} & \frac{-0.176e^{-0.48s}}{(6.9s+1)^2} \end{bmatrix},$$

$$G_2(s) = \begin{bmatrix} \frac{-2.368e^{-27.33s}}{33.3s+1} & \frac{-11.3e^{-3.79s}}{(21.74s+1)^2} \\ \frac{0.422e^{-8.72s}}{(250s+1)^2} & \frac{5.24e^{-60s}}{400s+1} \\ \frac{0.513e^{-s}}{s+1} & \frac{-0.33e^{-0.68s}}{(2.38s+1)^2} \\ \frac{15.54e^{-s}}{s+1} & \frac{4.48e^{-0.52s}}{11.11s+1} \end{bmatrix}.$$

An interaction analysis of this process involving a combination of the ERGA and the NI is performed in [37] with the conclusion that the most desirable decentralized pairing is y_1–u_4, y_2–u_2, y_3–u_1, y_4–u_3. The static RGA and the Σ_2 also suggest the same pairing as the ERGA and NI combination. For the Σ_2 this is not surprising since this measure can be given various energy interpretations similarly to the idea behind the

ERGA (c.f. [12] and [36]). The HIIA and the PM suggest the pairing y_1–u_4, y_2–u_1, y_3–u_2, y_4–u_3. However, the sum of the HIIA and PM elements for the pairing y_1–u_4, y_2–u_2, y_3–u_1, y_4–u_3 are very close to the sums for the suggested pairing.

In this example the sampling period was selected to 0.01 s and the LQII (for $q_u = 1 \cdot 10^{-9}$) was computed for all $4! = 24$ possible decentralized pairing combinations. The pairings y_1–u_1, y_2–u_4, y_3–u_2, y_4–u_3 and y_1–u_1, y_2–u_2, y_3–u_4, y_4–u_3 give the lowest LQII: 1.7999 and 1.8000. Since element $(2, 4)$ of the static RGA is negative, and such pairings should be avoided, the first LQII pairing recommendation is preferably rejected. The pairing suggestion from the HIIA and the PM gives a LQII of 1.8110 and the suggestion from the RGA and the Σ_2 gives a LQII of 1.8144. The LQII (for $q_u = 1$) gives lower values of the LQII (1.0459 for the recommended pairing y_1–u_1, y_2–u_4, y_3–u_2, y_4–u_3) due to a larger sum of variances for the full MIMO structure.

The largest ILQIA sum (0.5862) is obtained for pairing y_1–u_1, y_2–u_2, y_3–u_4, y_4–u_3. The second largest sum (0.4721) is for the same pairing as the RGA, the Σ_2 and the combination of ERGA and NI recommend.

In this example it is clear that there are several pairing candidates that have the potential to perform more or less equally according to the studied interaction measures. In particular, this holds when the expected control performance is evaluated in terms of output variance, as is the case for the LQII. The difference between the candidates is larger for ILQIA, which much clearer gives one single pairing candidate than the other measures.

6. Conclusions

Two new input-output pairing strategies based on LQG control, denoted LQII and ILQIA, have been presented. The obtained decentralized input-output pairing suggestions for different type of MIMO plants, taken from the literature, have been compared with those previously obtained with other interaction measures. The conclusions from the examples are summarized in Table 6. It was found that both of the proposed pairing strategies give suitable decentralized pairing suggestions for all the studied processes.

Table 6: Result of different pairing strategies for the studied examples.

	Correct or perhaps correct	Not correct	Comments
Ex 1	LQII, ILQIA, RGA, HIIA, PM, Σ_2		2×2 quadruple tank
Ex 2	LQII, ILQIA, DRGA, ERGA, Σ_2	RGA, HIIA, PM	Large differences in time delays
Ex 3	LQII, ILQIA, ARGA, RoMA	RGA^1, Σ_2^1, HIIA, PM	Only different time delays
Ex 4	LQII^2, ILQIA^2, RGA, HIIA^2		Bad performance of decentralized controller verified in simulations
Ex 5	(HIIA+RGA), (PM+RGA), (Σ_2+RGA), (LQII+NI+RGA)^2, (ILQIA+NI+RGA)^2	RGA, HIIA, PM, Σ_2, LQII^2, ILQIA^2	3×3 with the same non-minimum-phase dynamics
Ex 6	RGA, Σ_2, (ERGA+NI), HIIA, PM, (LQII+RGA)^3, ILQIA^3		4×4 distillation column ILQIA more conclusive

$^{1)}$ Inconclusive
$^{2)}$ MIMO control recommended
$^{3)}$ Many pairings expected to perform equally well

In the calculation of the LQII, only decentralized control structures have been considered. However, this is not an inherent limitation of this pairing strategy since it is able to evaluate the performance of other control structures as well. This is an advantage compared to the RGA, for example. Furthermore, since the LQII is based on what control performance that can be expected to be achieved with the designed control structures, this measure is easy to interpret. By definition, the index also quantifies how much better the performance can be with a full MIMO controller.

The ILQIA considers the maximum optimal integral feedback gain and thereby focuses on load disturbance rejection and low frequency behaviour of the system. It is also able to both give decentralized pairing suggestions and suggestions of multivariable controller structures. An advantage compared to the LQII is that it does not suffer from a severe computational burden when the number of inputs and outputs grow large.

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