Mathematical Morphology and Distance Transforms

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About the teacher

- PhD student at CBA
- Background in mathematics
- Research topic: Mathematical morphology and Discrete geometry
    - Distance measures in digital spaces
    - Adaptive mathematical morphology
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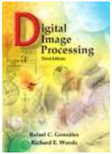
Today's lecture

- Mathematical morphology
- Distance transforms

- Chapters 9.1 – 9.6 in Gonzales-Woods book (for mathematical morphology)
- Notes for Distance transforms



Part 1

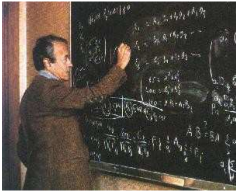
- Mathematical Morphology

Morphology in Britannica encyclopedia

- Morphology, in biology, is the study of the size, shape, and structure of animals, plants, and micro-organisms and the relationships of their internal parts
- Morphology, in linguistics, is the study of the internal construction of words
- Mathematical morphology?

History

- Mathematical Morphology was founded in the mid-sixties in France
- George Matheron and Jean Serra are two founders of mathematical morphology
- Study of geometry of porous media
- 11th International Symposium on Mathematical Morphology ISMM'2013, May 27-29, Uppsala (http://ismm.cb.uu.se/)



Mathematical morphology

- The theory for the analysis of spatial structures
- Analysis of shapes and objects
- It is based on:
  - Discrete space: Set theory, Lattice algebra
  - Continuous space: Partial differential equations
- Structuring elements

Set theory

- The language of mathematical morphology is set theory
- We will mostly work in $\mathbb{Z}^2$
  - Easy to extend to $\mathbb{Z}^n$
  - Can be extended to a continuous domain
- If $x = (x_1, x_2)$ is an element in X: $x \in X$
- Today's lecture covers only binary mathematical morphology (gray-scale mathematical morphology in Image Analysis 2)

Some set theory

- *Empty set*: $\emptyset$
- Every element in A is also in B (subset): $A \subset B$
- *Union* of A and B: $C = A \cup B = \{x | x \in A \text{ or } x \in B\}$
- *Intersection* of A and B: $C = A \cap B = \{x | x \in A \text{ and } x \in B\}$
- *Disjoint/mutually exclusive*: $A \cap B = \emptyset$

Some more set theory

- *Complement* of A: $A^C = \{x | x \notin A\}$
  (OBS! with respect to which universe)
- *Difference* of A and B: $A \setminus B = \{x | x \in A \text{ and } x \notin B\} = A \cap B^C$
- *Reflection (Transposition)* of A: $\hat{A} = \{-a | a \in A\}$
  The set A is symmetric if and only if $A = \hat{A}$
- *Translation* of A by a vector $z = (z_1, z_2)$:
  $(A)_Z = \{x | x = a + z,\ a \in A\}$

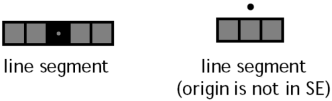
How to describe the SE

- Small set used to probe the image under study

Possible to define in many different ways!

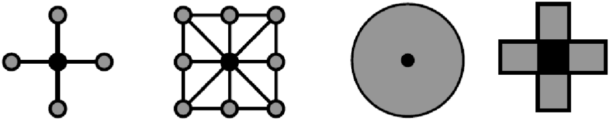
Information needed:

- Position of origin for SE
- Position of elements belonging to SE



How to describe the SE

- The shape and size must be adapted to geometric properties for the objects



OBS!

- Matlab assumes that the center of the structuring elements is its origin!
- Matlab: SE = strel(shape, parameters)

Five basic morphological operators

- ⊖ Erosion
- ⊕ Dilation
- ∘ Opening
- • Closing
- ⊗ Hit-or-Miss transform

$\ominus$ Erosion

- Does the structuring element fit the set?
- Erosion of a set X by structuring element B, $\varepsilon_B(X)$: all x in X such that B is in X when origin of B is x

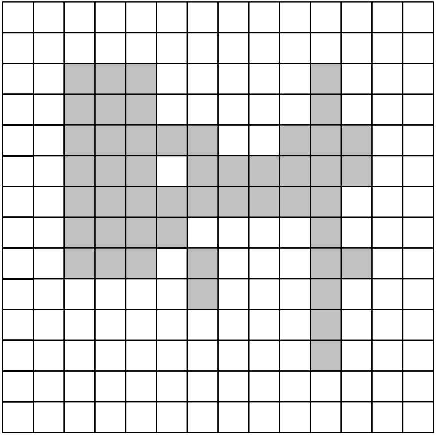
$$\varepsilon_B(X) = X \ominus B = \{x | B_x \subset X\}$$

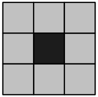
- Shrinks the object

OBS!

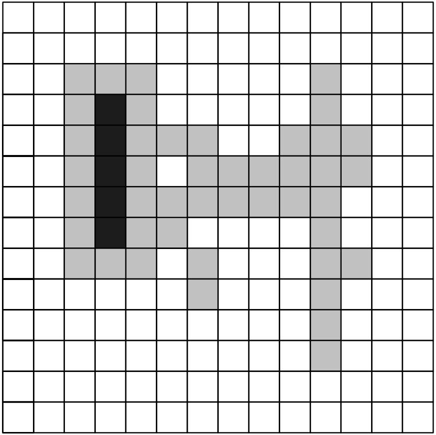
- Matlab: E = imerode(I,SE)

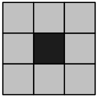
⊖ Erosion - problem





⊖ Erosion - result

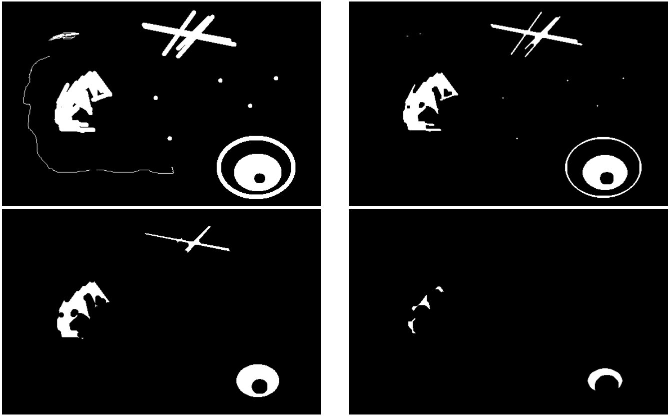




⊖ Erosion - Typical applications

Erosion

Removing structures of certain shape and size, given by SE



$\ominus$ Erosion - properties

1. It is increasing, i.e., if $A \subset C$, then $A \ominus B \subset C \ominus B$
2. If the origin belongs to the structuring element B, then the erosion is anti-extensive, i.e., $A \ominus B \subset A$
3. It is distributive over set intersection, i.e., $(A_1 \cap A_2) \ominus B = (A_1 \ominus B) \cap (A_2 \ominus B)$
4. It is translation invariant

$\oplus$ Dilation

- Does the structuring element hit the set?
- Dilation of a set X by structuring element B, $\delta_B(X)$: all x in X such that the reflection of B hits X when origin of B is x

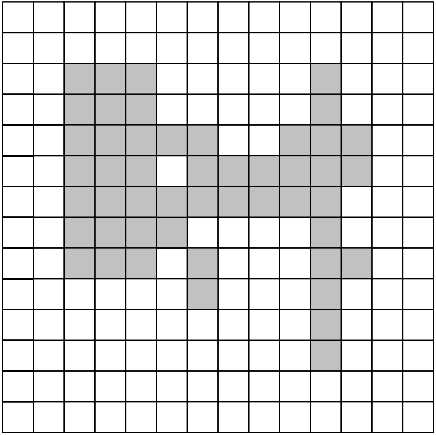
$$\delta_B(X) = X \oplus B = \{x | (\hat{B})_x \cap X \neq \emptyset\}$$

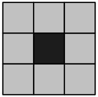
- Enlarges the object

OBS!

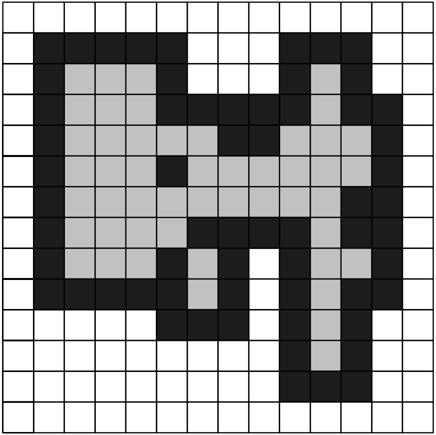
- Matlab: D = imdilate(I,SE)

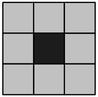
⊕ Dilation - problem





⊕ Dilation - result

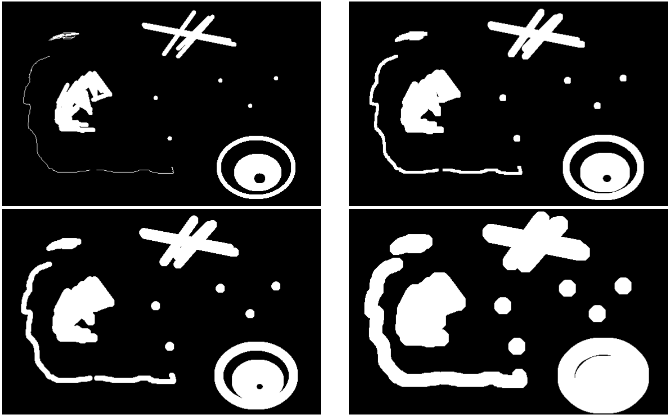




⨁ Dilation - Typical applications

Dilation

Filling of holes of certain shape and size, given by SE



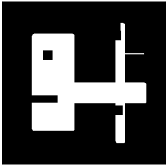
$\oplus$ Dilation - properties

1. It is increasing, i.e., if $A \subset C$, then $A \oplus B \subset C \oplus B$
2. If the origin belongs to the structuring element B, then it is extensive $A \subset A \oplus B$
3. It is distributive over set union, i.e., $(A_1 \cup A_2) \oplus B = (A_1 \oplus B) \cup (A_2 \oplus B)$
4. It is translation invariant

Erosion - Dilation duality

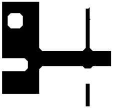
Erosion and dilation are dual with respect to complementation and reflection

$$(A \ominus B)^C = A^C \oplus \hat{B}$$

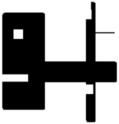


(a) A

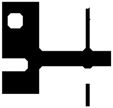
(b) $A \ominus B$



(c) $(A \ominus B)^C$



(d) A^C



(e) $A^C \oplus \hat{B}$

How to choose appropriate SE?

- Different structuring elements give different results
- Depend on previous knowledge
- Linear structuring elements for linear structures
- Circular structuring elements for circular structures

Opening and closing

- Combine erosion and dilation using the same SE
  - Opening
    Matlab: O = imopen(I,SE)
  - Closing
    Matlab: C = imclose(I,SE)

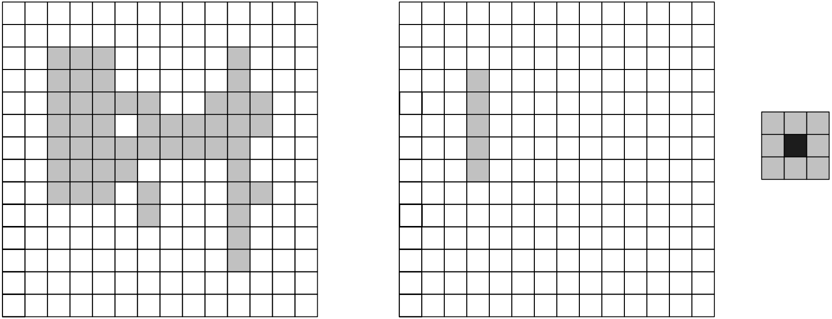
Opening

- Erosion followed by dilation

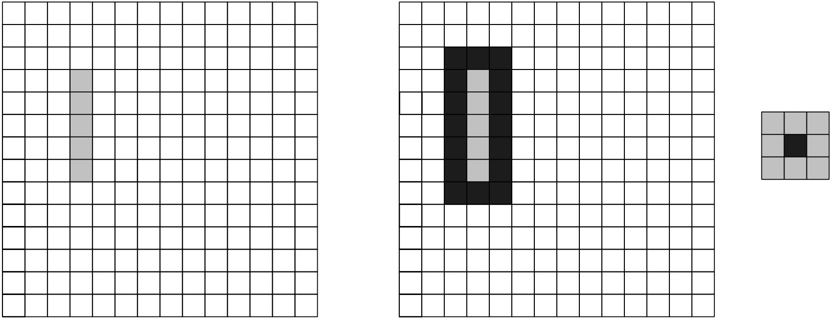
$$A \circ B = (A \ominus B) \oplus B$$

  - Eliminates protrusions
  - Breaks connections
  - Smooths contour

Opening (first erosion)



Opening (then dilation)



◦ Opening - properties

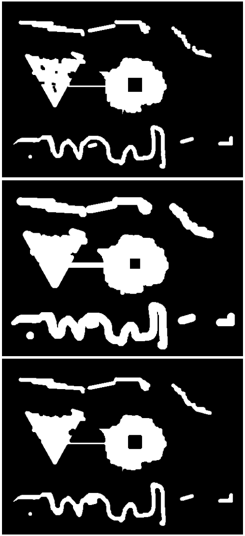
1. It is idempotent, i.e., $(A \circ B) \circ B = A \circ B$
2. It is increasing, i.e., if $A \subset C$, then $A \circ B \subset C \circ B$
3. It is anti-extensive, i.e., $A \circ B \subset A$
4. It is translation invariant

Closing

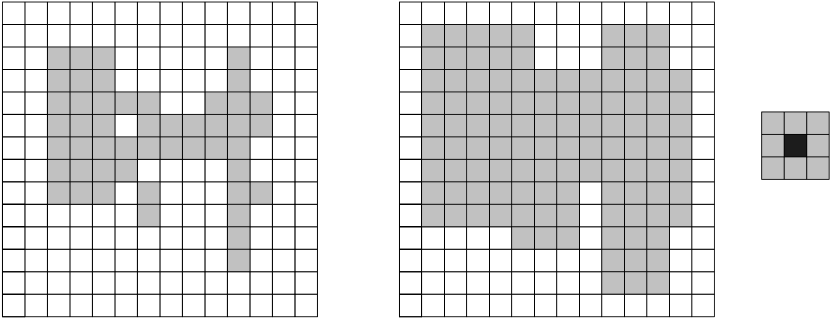
- Dilation followed by erosion

$$A \bullet B = (A \oplus B) \ominus B$$

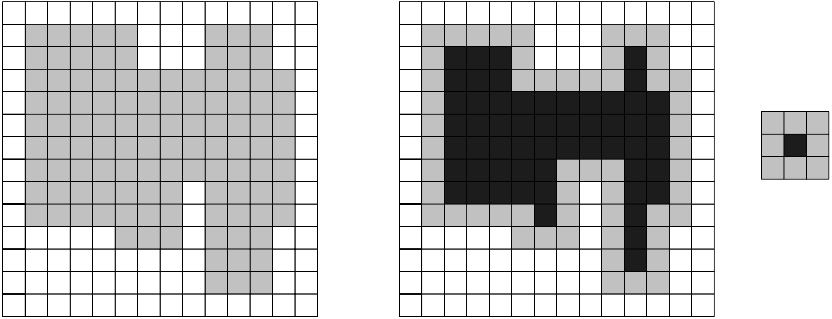
  - Smooths contour
  - Fuses narrow breaks and long thin gulfs
  - Eliminates small holes
  - Fills gaps in the contour



Closing (first dilation)



Closing (then erosion)



• Closing - properties

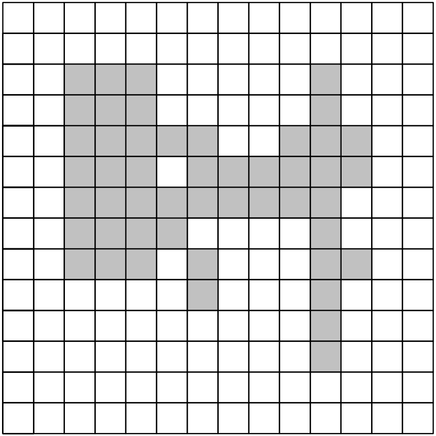
1. It is idempotent, i.e., $(A \bullet B) \bullet B = A \bullet B$
2. It is increasing, i.e., if $A \subseteq C$, then $A \bullet B \subseteq C \bullet B$
3. It is extensive, i.e., $A \subseteq A \bullet B$
4. It is translation invariant

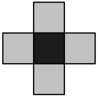
Opening - Closing duality

- Opening and closing are dual operators $A \bullet B = (A^c \circ \hat{B})^c$
- Opening (roll ball inside of the object)
  Boundary of $A \circ B$ is equal to points in B that reaches closest to the boundary of A, when B is rolled inside A
- Closing (roll ball outside of the object)
  Boundary of $A \bullet B$ is equal to points in B that reaches closest to the boundary of A, when B is rolled outside A

Exercise 1

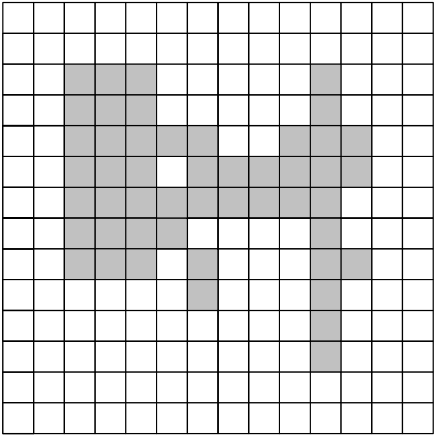
- Find erosion, dilation, opening and closing for the given image and SE

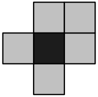




Exercise 2

- Find erosion, dilation, opening and closing for the given image and SE





$\otimes$ Hit-or-miss transformation

- Transformation that involves two structuring elements
- First has to fit with the object while, simultaneously, the second has to fit the background
- First has to hit the object while, simultaneously, second has to miss it

$$A \otimes B = (A \ominus B_1) \cap (A^C \ominus B_2)$$

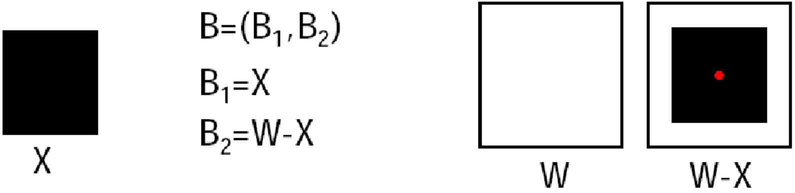
- Composite SE $B = (B_1, B_2)$: Object part (B_1) and background (B_2)

Hit-or-miss transformation ($\otimes$ or HMT)

- Alternative:

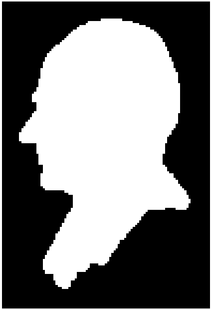
$$\begin{aligned} A \otimes B &= (A \ominus B_1) \cap (A^C \ominus B_2) \\ &= (A \ominus B_1) \cap (A \oplus \hat{B}_2)^C \\ &= (A \ominus B_1) \setminus (A \oplus \hat{B}_2) \end{aligned}$$

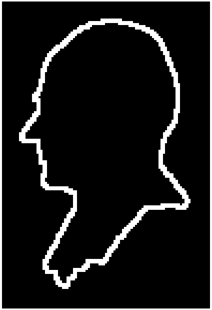
- B_1 and B_2 share the same origin and are disjoint sets



Applications of mathematical morphology

- More in the course book: Chapter 9.5
- Boundary extraction $\beta(A) = A \setminus (A \ominus B)$
- Boundary extraction $\beta(A) = (A \oplus B) \setminus A$





Part 2

- Distance Transforms

Distance transforms

Input: Binary image.
Output: In each object (background) pixel, write the distance to the closest background (object) pixel.

Definition

A function d is a metric (distance measure) for the pixels x, y, and z if

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$

Different metrics

Minkowski distances

Euclidean $d_E(p, q) = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}$
City block $d_4(p, q) = |x_p - x_q| + |y_p - y_q|$
Chess-board $d_8(p, q) = \max(|x_p - x_q|, |y_p - y_q|)$
where $p = (x_p, y_p)$ and $q = (x_q, y_q)$.

$\sqrt{2}$	1	$\sqrt{2}$
1	p	1
$\sqrt{2}$	1	$\sqrt{2}$

2	1	2
1	p	1
2	1	2

1	1	1
1	p	1
1	1	1

Different mask for distance transforms

Minkowski distances

Euclidean mask:

$\sqrt{2}$	1	$\sqrt{2}$
1	p	

City block mask:

2	1	2
1	p	

Chess-board mask:

1	1	1
1	p	

Weighted measures

4	3	4
3	p	

If distance between two 4-connected pixels is 3, then the distance between 4-connected pixels should be 4. Chamfer$\langle 3, 4 \rangle$ since $4/3 \approx 1.33$ is close to $\sqrt{2}$.

Matlab

DT = bwdist(I,method)

Algorithm for distance transformation

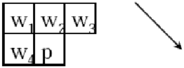
Distance from each object pixel to the closest background pixel

p current pixel

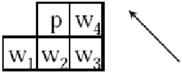
$g_1 - g_4$ neighbouring pixels

$w_1 - w_4$ weights (according to choice of metric)

1. Set background pixels to zero and object pixels to infinity.
2. Forward pass, from (0, 0) to $(\max(x), \max(y))$:
   if $p > 0$, $p = \min(g_i + w_i)$, $i = 1, 2, 3, 4$.



3. Backward pass, from $(\max(x), \max(y))$ to (0, 0):
   If $p > 0$, $p = \min(p, \min(g_i + w_i))$, $i = 1, 2, 3, 4$.



Chamfer $\langle 3, 4 \rangle$ distance

Binary original image

0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0
0	1	1	0	1	0	0	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	1	0
0	1	1	1	1	1	1	0
0	0	0	1	1	1	0	0
0	0	0	0	0	0	0	0

Chamfer $\langle 3, 4 \rangle$ distance

1. Starting image

0	0	0	0	0	0	0	0
0	∞	0	0	∞	0	0	0
0	∞	∞	0	∞	0	0	0
0	∞	∞	∞	∞	∞	∞	0
0	∞	∞	∞	∞	∞	∞	0
0	∞	∞	∞	∞	∞	∞	0
0	0	0	∞	∞	∞	0	0
0	0	0	0	0	0	0	0

4	3	4
3	p	

Chamfer $\langle 3, 4 \rangle$ distance

2. First pass from top left down to bottom right

0	0	0	0	0	0	0	0
0	3	0	0	3	0	0	0
0	3	3	0	3	0	0	0
0	3	4	3	4	3	3	0
0	3	6	6	7	6	4	0
0	3	6	9	10	8	4	0
0	0	0	3	6	8	0	0
0	0	0	0	0	0	0	0

	p	3
4	3	4

Chamfer $\langle 3, 4 \rangle$ distance

3. Second pass from bottom right down to top left

0	0	0	0	0	0	0	0
0	3	0	0	3	0	0	0
0	3	3	0	3	0	0	0
0	3	4	3	4	3	3	0
0	3	6	6	7	6	3	0
0	3	3	4	6	4	3	0
0	0	0	3	3	3	0	0
0	0	0	0	0	0	0	0

Applications using the distance transform (DT)

I. Find the shortest path between two points *a* and *b*.
   1. Generate the DT with *a* as the object.
   2. Go from *b* in the steepest gradient direction.

II. Find the radius of a round object
   1. Generate the DT of the object.
   2. The maximum value equals the radius.

- Next lecture and Lab 2: watershed algorithm!

Applications using the distance transform (DT)

III . Skeletons

Definitions: If O is the object, B is the background, and S is the skeleton, then

- S is topological equivalent to O
- S is centered in O
- S is one pixel wide (difficult!)
- O can be reconstructed from S

Matlab

SK = bwmorph(I,operation)

Skeletons (Centers of Maximal Discs)

A disc is made of all pixels that are within a given radius r. A disc in an object is *maximal* if it is not covered by any other disc in the object. A reversible representation of an object is the set of centers of maximal discs.

Algorithm

Find the skeleton with Centers of Maximal Discs (CMD)

Completely reversible situation

1. Generate distance transform of object
2. Identify CMDs (smallest set of maxima)
3. Link CMDs

"Pruning" is to remove small branches (no longer fully reversible.)

Topic not covered in this lecture

- Mathematical morphology for gray-valued image (Image Analysis 2)
- Morphological Skeletons
- Number of applications of mathematical morphology and distance transform (Lab 2)

Trends in mathematical morphology- possible topics for master thesis

- Adaptive mathematical morphology
- Nonlocal mathematical morphology
- Multi-valued mathematical morphology
- Efficient GPU implementations

Exercise

- Reading material: Chapters 9.1 - 9.6
- Two problem given during the lecture (OBS! non-symmetric SE)
- Book: Exercise 9.3, 9.5, 9.7, 9.14, 9.19