Uppsala University
Department of Information Technology
Systems and Control
Prof Torsten Söderström

Control Design, Dec 17, 2004 — Answers and brief solutions

Problem 1

(a) In this case

$$G(0) = \begin{pmatrix} 1/2 & 10 \\ 1/5 & 5/3 \end{pmatrix}.$$

Hence

$$\begin{aligned}
\text{RGA}(G(0)) &= \begin{pmatrix} 1/2 & 10 \\ 1/5 & 5/3 \end{pmatrix} \odot \frac{1}{5/6-2} \begin{pmatrix} 5/3 & -1/5 \\ -10 & 1/2 \end{pmatrix} \\
&= \frac{-6}{7} \begin{pmatrix} 5/6 & -2 \\ -2 & 5/6 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -5 & 12 \\ 12 & -5 \end{pmatrix}
\end{aligned}$$

(b) Avoid pairing leading to negative diagonal elements in RGA. Hence combine u_1 with y_2, and u_2 with y_1.

Problem 2

(a) If x_1 is the unperturbed output and $x_2 = W_S(Gu + w)$, the following model is obtained:

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w
\end{aligned}$$

(b)

$$D^T D = 1 \Longrightarrow \text{invertible!}$$

$$\det(\lambda I - A + NC) = \det\left(\begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda \end{pmatrix}\right) = \lambda(\lambda+1) = 0$$

$\Longrightarrow A - NC$ has no eigenvalues in RHP. The regulator is therefore determined using the standard scheme.

$$A^T S + SA + M^T M - SBB^T S = 0 \Longrightarrow \begin{cases} 2(s_{12} - s_{11}) + 1 - s_{11}^2 = 0 \\ s_{22} - s_{12} - s_{11}s_{12} = 0 \\ 1 - s_{12}^2 = 0 \end{cases}$$

$$\Longrightarrow S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Longrightarrow L = \begin{pmatrix} 1 & 0 \end{pmatrix} S = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$F_y(p) = L\left(pI - A + BB^T S + NC\right)^{-1} N = \frac{p+1}{p(p+2)}$$

Problem 3

(a) Laplace transforming the state space equations give directly

$$\begin{aligned} X_1(s) &= \frac{1}{s+1}U_1(s), \quad X_2(s) = \frac{2}{s+2}U_2(s), \\ X_3(s) &= \frac{3}{s+1}U_1(s), \quad X_4(s) = \frac{-2}{s+1}U_2(s) \end{aligned}$$

and then

$$Y_1(s) = X_1(s) + X_2(s), \quad Y_2(s) = X_3(s) + X_4(s)$$

which directly leads to $Y(s) = G(s)U(s)$.

(b) The minors are

$$\frac{1}{s+1}, \ \frac{2}{s+2}, \ \frac{3}{s+1}, \ \frac{-2}{s+1}, \ \frac{-2}{(s+1)^2} - \frac{6}{(s+1)(s+2)}$$

The least common denominator is $(s+1)^2(s+2)$ which is the pole polynomial.

(c) It follows from part (b) that S2 must have model order $= 3$. y_2 is affected by the sum $x_3 + x_4$, and the states x_3 and x_4 have the same pole. Hence try as a state vector for S2

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_4 \end{pmatrix}$$

This gives

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = -x_1 + u_1 = -z_1 + u_1 \\ \dot{z}_2 &= \dot{x}_2 = -2x_2 + u_2 = -2z_2 + u_2 \\ \dot{z}_3 &= \dot{x}_3 + \dot{x}_4 = (-x_3 + 3u_1) + (-x_4 - 2u_2) \\ &= -(x_3 + x_4) + 3u_1 - 2u_2 \\ &= -z_3 + 3u_1 - 2u_2 \end{aligned}$$

We thus have

$$\begin{aligned} \dot{z} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & -2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} z \end{aligned}$$

Problem 4

(a) The Riccati equation gives

$$0 = AP + PA^T + \gamma^4 NN^T - PC^TCP$$

or

$$\begin{aligned}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+\gamma^4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \end{pmatrix}\end{aligned}$$

which leads to the equations

$$\begin{aligned} 0 &= 2p_{12} - p_{11}^2 \\ 0 &= p_{22} - p_{11}p_{12} \\ 0 &= \gamma^4 - p_{12}^2 \end{aligned}$$

As P must be positive definite, the solution is

$$P = \begin{pmatrix} \sqrt{2}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{2}\gamma^3 \end{pmatrix}$$

The Kalman gain becomes

$$K = PC^T = \begin{pmatrix} \sqrt{2}\gamma \\ \gamma^2 \end{pmatrix}$$

The Kalman filter will be

$$\hat{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\hat{x} + \begin{pmatrix} \sqrt{2}\gamma \\ \gamma^2 \end{pmatrix}(y - \hat{x}_1)$$

(b) The Riccati equation becomes in this case

$$\begin{aligned} 0 &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma^4 \end{pmatrix} \\ &- \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\end{aligned}$$

Simplifying the equation, and evaluating the different elements leads to the following system of equations

$$\begin{aligned} 0 &= 2p_{12} - p_{11}^2 - \frac{1}{\gamma^2}p_{12}^2 \\ 0 &= p_{22} - p_{11}p_{12} - \frac{1}{\gamma^2}p_{12}p_{22} \\ 0 &= \gamma^4 - p_{12}^2 - \frac{1}{\gamma^2}p_{22}^2 \end{aligned}$$

It is straightforward to see that the claimed solution does indeed satisfy these three equations.

The Kalman gain becomes

$$K = PC^T R_2^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} \sqrt{3}\gamma & 1 \\ \gamma^2 & \sqrt{3}\gamma \end{pmatrix}$$

The Kalman filter can be written elementwize as

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + 0.5\sqrt{3}\gamma(y_1 - \hat{x}_1) + 0.5(y_2 - \hat{x}_2) \\ \dot{\hat{x}}_2 &= 0.5\gamma^2(y_1 - \hat{x}_1) + 0.5\sqrt{3}(y_2 - \hat{x}_2) \end{aligned}$$

Problem 5

(a) The highest possible cut-off frequency is achieved when there is no phase margin at all. Determine ω_c from $\arg G(i\omega_c) = -\pi$. This leads to

$$-\pi = \arg(1 - i\omega_c) - 2\arg(1 + i\omega_c) = -\arctan(\omega_c) - 2\arctan(\omega_c)$$

and hence

$$3\arctan(\omega_c) = \pi \Rightarrow \arctan(\omega_c) = \pi/3 \Rightarrow \omega_c = \sqrt{3}$$

(b) Determine K from

$$|G(i\omega_c)| \leq 1 \Rightarrow \frac{K}{\sqrt{\omega_c^2 + 1}} \leq 1 \Rightarrow K \leq 2$$

(c) The analysis in part (a) modifies to

$$-\pi + \pi/4 = \arg(1 - i\omega_c) - 2\arg(1 + i\omega_c) = -\arctan(\omega_c) - 2\arctan(\omega_c)$$

and hence

$$3\arctan(\omega_c) = 3\pi/4 \Rightarrow \arctan(\omega_c) = \pi/4 \Rightarrow \omega_c = 1$$

The analysis in part (b) becomes

$$|G(i\omega_c)| \leq 1 \Rightarrow \frac{K}{\sqrt{\omega_c^2 + 1}} \leq 1 \Rightarrow K \leq \sqrt{2}$$

Problem 6

(a) The Riccati equation becomes

$$\begin{aligned} 0 &= A^T S + SA + Q_1 - SBQ_2^{-1}B^T S \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\rho^4} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

which leads to the following system of equations

$$\begin{aligned} 0 &= 1 - s_{12}^2/\rho^4 \\ 0 &= s_{11} - s_{12}s_{22}/\rho^4 \\ 0 &= 2s_{12} - s_{22}^2/\rho^4 \end{aligned}$$

The solution is readily found to be

$$S = \begin{pmatrix} \sqrt{2}\rho & \rho^2 \\ \rho^2 & \sqrt{2}\rho^3 \end{pmatrix}$$

The corresponding feedback vector is

$$\begin{aligned} L &= \frac{1}{\rho^4} B^T S \\ &= \frac{1}{\rho^4} \begin{pmatrix} s_{12} & s_{22} \end{pmatrix} \\ &= \begin{pmatrix} \rho^2 & \sqrt{2}\rho^3 \end{pmatrix} / \rho^4 \end{aligned}$$

(b) The characteristic equation for the closed loop system is

$$\begin{aligned} 0 &= \det(sI - A + BL) \\ &= \det \begin{pmatrix} s & -1 \\ \rho^{-2} & s + \sqrt{2}\rho^{-1} \end{pmatrix} = s^2 + \sqrt{2}\rho^{-1}s + \rho^{-2} \end{aligned}$$

which has its solutions in

$$s = \frac{\sqrt{2}}{2}(-1 \pm i)\frac{1}{\rho}$$

(c) The closed loop system is given by the state space model

$$\dot{x} = (A - BL)x + Nv$$

or more explicitly,

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

Let the process noise have unit spectrum. The covariance matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

is found from the Lyapunov equation

$$0 = \begin{pmatrix} 0 & 1 \\ -\rho^{-2} & -\sqrt{2}\rho^{-1} \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & -\rho^{-2} \\ 1 & -\sqrt{2}\rho^{-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Evaluating the elements leads to the following system of equations

$$\begin{aligned} 0 &= 2p_{12} \\ 0 &= p_{22} - \rho^{-2}p_{11} - \sqrt{2}\rho^{-1}p_{12} \\ 0 &= -2\rho^{-2}p_{12} - 2\sqrt{2}\rho^{-1}p_{22} + 1 \end{aligned}$$

The solution to these equations are found to be

$$P = \frac{1}{2\sqrt{2}} \begin{pmatrix} \rho^3 & 0 \\ 0 & \rho \end{pmatrix}$$