Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Control Design

Date: December 17, 2004

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1, you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik –. Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are **not allowed**: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!

Problem 1

Consider a two-input, two-output system with the transfer function

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix}$$

(a) Determine RGA($G(0)$). **4 points**

(b) Which input-output pairing should be preferred? **2 points**

Problem 2

Assuming no disturbances are present, the following model is assumed to describe a dynamic system.

$$\frac{d}{dt}y(t) + y(t) = u(t).$$

The system should be controlled by a $\mathcal{H}_2$ designed regulator, where the following weights apply:

$$W_S(s) = \frac{1}{s}, \quad W_T(s) = 1 \quad \text{och} \quad W_u = 1.$$

(a) Determine a state space model for the extended system in the standard form

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nw(t) \\ z(t) &= Mx(t) + Du(t) \\ y(t) &= Cx(t) + w(t) \end{aligned}$$

4 points

(b) Determine the transfer function $F_y(p)$, where the resulting regulator is $u(t) = -F_y(p)y(t)$ **4 points**

Problem 3

(a) Consider a multivariable system with the transfer function

$$G(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \\ \frac{3}{s+1} & \frac{-2}{s+1} \end{pmatrix}$$

Show that the system can be represented in state space form as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \\ 0 & -2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} x \end{aligned}$$

3 points

(b) What is the pole polynomial of the transfer function $G(s)$ given in part (a)? **2 points**

(c) Consider the state space representation, say S1, in part (a). Show how it can be reduced to another state space representation, say S2. The new representation S2 should have smaller order than S1, and be both controllable and observable. **4 points**

Problem 4

Consider a rocket in space, which we model as a double integrator for the movement in one direction. The acceleration is due to some process noise, so with the state variables $x_1 = y,\ x_2 = \dot{y}$ the state space model is

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + e
\end{aligned}
$$

We assume for the time being that the position y is measured with some error e.

Assume that the spectra of the noise sources are

$$\phi_v = R_1 = \gamma^4, \quad \phi_e = R_2 = 1$$

(a) Determine the Kalman filter. Determine also the covariance matrix of the estimation error

$$P = E\tilde{x}(t)\tilde{x}^T(t)$$

4 points

(b) Assume next that a second sensor is added, so that also the velocity $\dot{y}$ is measured. The output equation is then changed to

$$y = x + e$$

where now e is a two-dimensional vector with spectrum

$$\phi_e = R_2 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 \end{pmatrix}$$

Determine the optimal Kalman filter in this case. Show that in this case the covariance matrix of the estimation error is

$$P = E\tilde{x}(t)\tilde{x}^T(t) = \frac{1}{2} \begin{pmatrix} \sqrt{3}\gamma & \gamma^2 \\ \gamma^2 & \sqrt{3}\gamma^3 \end{pmatrix}$$

3 points

Problem 5

(a) Consider the system

$$G(s) = \frac{K(1-s)}{(s+1)^2}$$

What is the utmost highest possible cut-off frequency that can be chosen when controlling this system with a feedback $u(t) = -y(t)$? **3 points**

(b) How high can the gain K be chosen, if the system is controlled with unit feedback ($u = -y$) before losing stability? **2 points**

(c) Assume that it is imposed in the design that there must be a phase margin of at least 45^o degrees. How does that effect the questions treated in parts (a) and (b)? **4 points**

Problem 6

Consider a double integrator. The acceleration is due both to an input u and to some process noise, so with the state variables $x_1 = y$, $x_2 = \dot{y}$ the state space model is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

(a) Assume that the aim of control is to minimize the variance of the position $y = x_1$. Consider first the noise free case and find a full state feedback controller, $u(t) = -Lx(t)$, so that the criterion

$$V = \int_0^\infty [x_1^2(t) + \rho^4 u^2(t)]dt$$

is minimized. **4 points**

(b) Determine the closed loop poles when the state feedback from (a) is applied. **3 points**

(c) Let the feedback from part (a) be applied. Determine the covariance matrix of the state vector, that is compute

$$P = Ex(t)x^T(t)$$

4 points