Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Control Design (Reglerteknisk design, 1TT492)

Date: December 22, 2005

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik –. Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are **not allowed**: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!

Problem 1

Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 3 & 0 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} x
\end{aligned}
$$

(a) Is the system controllable? **2 points**

(b) Is the system observable? **2 points**

(c) Determine the poles of the system. **2 points**

Problem 2

Use internal model control with λ-tuning to design a regulator for a DC motor with the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

(a) Express the regulator using λ as a parameter. Is the regulator integrating? **2 points**

(b) Determine the loop gain? **2 points**

(c) Choose λ so that the controlled system has cut-off frequency ω_c. **2 points**

(d) Determine the phase margin! **2 points**

Problem 3

Let $x(t)$ be a stationary continuous-time process with covariance function $r_x(\tau)$ and spectral density $\phi_x(\omega)$.

(a) Consider the process

$$y(t) = \frac{x(t+h) - x(t-h)}{2h}$$

which is an approximate time-derivative of $x(t)$.

Show that the covariance function $r_y(\tau)$ of $y(t)$ is

$$r_y(\tau) = \frac{2r_x(\tau) - r_x(\tau + 2h) - r_x(\tau - 2h)}{4h^2}$$

3 points

(b) Show that the spectrum of $y(t)$ is

$$\phi_y(\omega) = \phi_x(\omega)\frac{\sin^2(\omega h)}{h^2}$$

3 points

(c) What is the spectrum of the true derivative $\dot{x}(t)$? **1 point**

Problem 4

Consider an unstable system with transfer function

$$G(s) = \frac{1}{1 - sT}, \qquad (T > 0)$$

Assume that it is to be controlled with an LQ regulator minimizing the criterion

$$V = \int_0^{\infty} [y^2(t) + \rho u^2(t)]dt, \qquad (\rho > 0)$$

(a) Determine the optimal regulator $F_y(s)$. **4 points**

(b) Determine the loop gain $L_o(s) = G(s)F_y(s)$. **2 points**

(c) Determine the cutoff frequency ω_c defined by $|L_o(i\omega)| = 1$. When ρ is varied, what is the smallest value ω_c can take? **4 points**

Problem 5

(a) Consider a harmonic oscillator with transfer function

$$G(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}$$

Represent it in state space form as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -\omega_o^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_o^2 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

Determine the state space form of the sampled version of this system, when the sampling interval equals h. **3 points**

(b) Determine the pulse transfer function ('överföringsoperatorn') of the sampled system of part (a). **3 points**

(c) Given a discrete-time system with the pulse transfer function

$$H(q) = \frac{1}{q + 1}$$

Can this system occur by sampling a second order system with the sampling interval being h? Motivate your answer! **3 points**

Problem 6

A certain SISO system is given by

$$y(t) = u(t - \tau) + w(t)$$

(u input, w process disturbance) that is, it consists of a pure time-delay with no additional dynamics. However, the time-delay τ is not precisely known, but the objective of the feedback system is to make the system robust against variations in the value of τ. Some experiments indicate that $\tau \approx 2$, which can be used for a nominal model. The design will be based on the following scheme.

1. First design an integrator as feedback,

$$G_I(s) = \frac{\beta}{s}$$

so that the system gets a reasonable phase margin.

2. Make the approximation of the process dynamics as

$$G_P(s) = e^{-s\tau} = \frac{1 - s\tau/2}{1 + s\tau/2}$$

and represent the loop gain $G_P G_I$ in state space form.

3. Design a H_∞ controller based on the Glover-McFarlane design principle.

Tasks to be solved:

a) Choose $\beta = 0.5$ and $\tau = 2$ (which will give the phase margin $\varphi_m \approx 33^o$) and give a state space realization in observable canonical form for the loop gain (see point 2 above). **2 points**

b) Why has a pure integrator, and not a PI-regulator, been chosen as the initial design? **2 points**

Hint. Check the assumptions for the Glover-McFarlane design method.

c) Verify that in the design of the robust controller the solutions to the pertinent Riccati equations are

$$Z = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}$$

Also, determine the matrix L in the feedback. **3 points**

d) When determining the matrix K the design parameter α is chosen as somewhat larger than 1. What will happen if one makes the choice $\alpha = 1$? **3 points**

Hint. XZ and ZX have the same eigenvalues.

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Control Design, December 22, 2005 — Answers and brief solutions

Problem 1

(a) The controllability matrix becomes

$$W_c = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 3 & 0 & -6 & 0 & 12 & 0 \end{pmatrix}$$

The first two rows are identical and therefore linearly dependent. The rank of W_c is 2. The system is not controllable.

(b) The beginning of the observability matrix is

$$W_o = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & -2 & -2 \\ & & \\ & & \end{pmatrix}$$

As the upper three rows are linearly independent, it is not necessary to compute further rows to conclude that W_o has rank 3. The system is observable.

(c) The A matrix has two eigenvalues in $s = -1$ and one in $s = -2$. As the system is not controllable, it is not possible to say that this is precisely the poles. Computing the transfer function gives

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \ldots = \frac{1}{s+1}\begin{pmatrix} 3 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{s+2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3(2s+3)}{(s+1)(s+2)} & \frac{3}{s+1} \\ \frac{3}{s+1} & \frac{3}{s+1} \end{pmatrix} \end{aligned}$$

The minors of order 1 are: $\frac{3(2s+3)}{(s+1)(s+2)}$ and $\frac{3}{s+1}$.
There is one minor of order 2: $\det G(s) = \frac{9}{(s+1)(s+2)}$.

The pole polynomial is hence $(s+1)(s+2)$. The system has one pole in $s = -1$ and one pole in $s = -2$.

Problem 2

(a)

$$Q(s) = \frac{1}{(1+\lambda s)^2} s(s+1)$$

$$\begin{aligned} F_y(s) &= \frac{Q(s)}{1-Q(s)G(s)} = \frac{s(s+1)}{(1+\lambda s)^2} \frac{1}{1 - \frac{s(s+1)}{(1+\lambda s)^2}\frac{1}{s(s+1)}} \\ &= s(s+1)\frac{1}{(1+\lambda s)^2 - 1} = \frac{s(s+1)}{\lambda^2 s^2 + 2\lambda s} = \frac{s+1}{\lambda^2 s + 2\lambda} \end{aligned}$$

As $F_y(0) \neq \infty$, the regulator is not integrating.

(b) The loop gain becomes

$$L(s) = F_y(s)G(s) = \frac{s+1}{\lambda^2 s + 2\lambda}\frac{1}{s(s+1)} = \frac{1}{\lambda^2 s^2 + 2\lambda s}$$

(c) The cut-off frequency ω_c is defined by $|L(i\omega_c)| = 1$. Hence,

$$|-\omega_c^2\lambda^2 + 2i\omega_c\lambda| = 1$$

Set $\alpha = \omega_c \lambda$. We have

$$|-\alpha^2 + 2i\alpha| = 1 \Rightarrow \alpha^4 + 4\alpha^2 - 1 = 0 \Rightarrow \alpha^2 = -2 + \sqrt{5} \Rightarrow \alpha = \sqrt{\sqrt{5}-2} \approx 0.486.$$

Hence we have

$$\lambda = \frac{0.486}{\omega_c}$$

(d) The phase margin is

$$\begin{aligned} \varphi_m &= 180^o + \arg L(i\omega_c) = 180^o + \arg\frac{1}{-\alpha^2 + 2i\alpha} \\ &= 180^o - \arg(-\alpha^2 + 2i\alpha) \\ &= \arctan\frac{2\alpha}{\alpha^2} = \arctan\frac{2}{\alpha} = \arctan(\frac{2}{\sqrt{\sqrt{5}-2}}) = \arctan(2\sqrt{\sqrt{5}+2}) \approx 76.34^o \end{aligned}$$

Problem 3

(a) The covariance function is easily found:

$$\begin{aligned} r_y(\tau) &= Ey(t+\tau)y(t) \\ &= \frac{1}{4h^2}E[x(t+\tau+h) - x(t+\tau-h)][x(t+h) - x(t-h)] \\ &= \frac{1}{4h^2}[r_x(\tau) - r_x(\tau+2h) - r_x(\tau-2h) + r_x(\tau)] \\ &= \frac{1}{4h^2}[2r_x(\tau) - r_x(\tau+2h) - r_x(\tau-2h)] \end{aligned}$$

(b) The spectrum becomes

$$\begin{aligned}
\phi_y(\omega) &= \int_{-\infty}^{\infty} r_y(\tau)e^{-i\omega\tau}d\tau \\
&= \frac{1}{4h^2}\int_{-\infty}^{\infty}[2r_x(\tau) - r_x(\tau+2h) - r_x(\tau-2h)]e^{-i\omega\tau}d\tau \\
&= \frac{1}{4h^2}\left[2\phi_x(\omega) - \int_{-\infty}^{\infty} r_x(\tau')e^{-i\omega\tau'+i\omega 2h}d\tau' - \int_{-\infty}^{\infty} r_x(\tau')e^{-i\omega\tau'-i\omega 2h}d\tau'\right] \\
&= \frac{1}{4h^2}\phi_x(\omega)\left[2 - e^{i2\omega h} - e^{-i2\omega h}\right] \\
&= \frac{1}{4h^2}\phi_x(\omega)\left[2 - 2\cos(2\omega h)\right] \\
&= \frac{1}{4h^2}\phi_x(\omega)\left[2 - 2\left\{1 - 2\sin^2(\omega h)\right\}\right] \\
&= \phi_x(\omega)\frac{\sin^2(\omega h)}{h^2}
\end{aligned}$$

(c)

$$\phi_{\dot{x}}(\omega) = |i\omega|^2\phi_x(\omega) = \omega^2\phi_x(\omega)$$

Note that $\phi_y(\omega) \approx \phi_{\dot{x}}(\omega)$ for small h (or small ω). Further, $\phi_y(\omega)$ and $\phi_{\dot{x}}(\omega)$ will differ significantly for large ω.

Problem 4

(a)

$$\begin{aligned}
\dot{x} &= \tfrac{1}{T}x - \tfrac{1}{T}u \\
y &= x
\end{aligned}$$

Riccati equation

$$0 = 2\frac{1}{T}S + 1 - \frac{S^2}{T^2}\frac{1}{\rho}, \qquad L = -\frac{S}{T\rho}$$

$$S^2 - 2ST\rho - T^2\rho = 0$$

with solution

$$\begin{aligned}
S &= T\rho \pm \sqrt{T^2\rho^2 + T^2\rho} \\
&= T\rho[1 + \sqrt{1 + 1/\rho}]
\end{aligned}$$

$$F_y(s) = L = -1 - \sqrt{1 + 1/\rho}$$

(b)

$$L_o(s) = \frac{L}{1 - sT}$$

(c)

$$\begin{aligned}
|L_o(i\omega_c)| &= 1 \;\Rightarrow \\
1+\omega_c^2T^2 &= |L|^2 = 1 + (1+1/\rho) + 2\sqrt{1+1/\rho} \\
\omega_c^2 &= \frac{1}{T^2}[1+1/\rho + 2\sqrt{1+1/\rho}] \geq \frac{3}{T^2} \text{ with equality for } \rho \to \infty
\end{aligned}$$

Problem 5

(a) Compute first the matrix exponential.

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1}\begin{pmatrix} s & -1 \\ \omega_o^2 & s \end{pmatrix}^{-1} = \mathcal{L}^{-1}\frac{1}{s^2+\omega_o^2}\begin{pmatrix} s & 1 \\ -\omega_o^2 & s \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \cos(\omega_o t) & \frac{1}{\omega_o}\sin(\omega_o t) \\ -\omega_o \sin(\omega_o t) & \cos(\omega_o t) \end{pmatrix}
\end{aligned}$$

Set

$$C = \cos(\omega_o h), \qquad S = \sin(\omega_o h)$$

Then

$$\begin{aligned}
F &= \begin{pmatrix} C & \frac{1}{\omega_o}S \\ -\omega_o S & C \end{pmatrix} \\
G &= \int_0^h e^{As}B ds = \int_0^h \begin{pmatrix} \omega_o \sin(\omega_o s) \\ \omega_o^2 \cos(\omega_o s) \end{pmatrix} ds = \begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
H(q) &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} q-C & -\frac{1}{\omega_o}S \\ \omega_o S & q-C \end{pmatrix}^{-1}\begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix} \\
&= \frac{1}{(q-C)^2+S^2}\begin{pmatrix} q-C & \frac{1}{\omega_o}S \end{pmatrix}\begin{pmatrix} 1-C \\ \omega_o S \end{pmatrix} \\
&= \frac{(1-C)(q+1)}{q^2-2qC+1}
\end{aligned}$$

(c) Use the results of parts (a) and (b). Consider a continuous-time system with the transfer function

$$G(s) = \frac{K\omega_o^2}{s^2+\omega_o^2}$$

(where the parameters K and ω_o are to be determined). The pulse transfer function of this system is, according to part (b)

$$H(q) = \frac{K(1-C)(q+1)}{q^2-2Cq+1}$$

Now choose in particular

$$K = 0.5, \quad \omega_o = \pi/h$$

Then, $C = -1$ and

$$H(q) = \frac{0.5 \times 2(q+1)}{q^2+2q+1} = \frac{1}{q+1}$$

Problem 6

a)

$$L(s) = G_P(s)G_I(s) = \frac{1-s}{1+s}\frac{0.5}{s} = \frac{-0.5s+0.5}{s^2+s}$$

In observable canonical form

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

b) Assume that the feedback (PI - regulator) $G_{PI}(s) = \frac{\alpha s+\beta}{s}$ is used. The loop gain becomes

$$L(s) = G_P(s)G_{PI}(s) = \frac{1-s}{1+s}\frac{\alpha s+\beta}{s}$$

which is not strictly proper. Hence a state space representation will have a nonzero direct term (a matrix $D \neq 0$), which violates the assumptions used in the Glover-McFarlane design procedure.

c) Straightforward calculations give

$$\begin{aligned} & A^TX + XA - XBB^TX + C^TC \\ = & \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \\ & - \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}\begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}\begin{pmatrix} -0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} -0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix} + \begin{pmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Similarly,

$$\begin{aligned} & AZ + ZA^T - ZC^TCZ + BB^T \\ = & \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \\ & - \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} + \begin{pmatrix} -0.5 \\ 0.5 \end{pmatrix}\begin{pmatrix} -0.5 & 0.5 \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix} - \begin{pmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{pmatrix} + \begin{pmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{pmatrix} \\ = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Further,

$$L = B^TX = \begin{pmatrix} -0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

d) Let λ_m denote the largest eigenvalue to XZ. It holds that

$$\begin{aligned} R &= I - \frac{1}{\gamma^2}(I + ZX) \\ &= \frac{1}{\gamma^2}(\gamma^2 I - I - ZX) \\ \gamma &= \alpha\sqrt{1 + \lambda_m} \\ K &= R^{-1}ZC^T \\ R^{-1} &= \gamma^2[(\gamma^2 - 1)I - ZX]^{-1} \end{aligned}$$

However, if $\alpha = 1$ it holds that $\gamma^2 = 1 + \lambda_m$ and

$$R^{-1} = (1 + \lambda_m)(\lambda_m I - ZX)^{-1}$$

Note now that (use that X and Z are symmetric) the matrix XZ has the same eigenvalues as $(XZ)^T = Z^T X^T = ZX$. Hence we find that $\lambda I - ZX$ is singular. We cannot compute neither R, nor K.