Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Control Design (Reglerteknisk design, 1TT492)

Date: December 21, 2006

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are **not allowed**: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!

Problem 1

Consider LQ control of the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

The criterion to be minimized is

$$V = \int [\alpha^2 y^2(t) + u^2(t)]dt, \qquad (\alpha > 0)$$

and hence

$$Q_1 = \alpha^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q_2 = 1$$

Determine the optimal feedback gain vector L. Determine also the loop gain $L(pI - A)^{-1}B$. **6 points**

Problem 2

Consider a triangular system with two inputs and two outputs given in the form

$$Y(s) = \begin{pmatrix} G_{11}(s) & 0 \\ G_{21}(s) & G_{22}(s) \end{pmatrix} U(s)$$

which is controlled by a triangular controller given as

$$U(s) = -\begin{pmatrix} F_{11}(s) & 0 \\ F_{21}(s) & F_{22}(s) \end{pmatrix} Y(s) + \begin{pmatrix} H_{11}(s) & 0 \\ H_{21}(s) & H_{22}(s) \end{pmatrix} R(s)$$

The aim is to get a closed loop system in diagonal form

$$Y(s) = \begin{pmatrix} C_{11}(s) & 0 \\ 0 & C_{22}(s) \end{pmatrix} R(s)$$

(a) Assume that the feedback is in diagonal form, so that $F_{21}(s) = 0$. Can one then choose the feedforward filter $H(s)$ so that the closed-loop system becomes diagonal? If yes, what is the sensitivity function in this case? **3 points**

(b) Assume that the feedforward link from the reference signal is in diagonal form, so that $H_{21}(s) = 0$. Can one then choose the feedback transfer function $F(s)$ so that the closed-loop system becomes diagonal? If yes, what is the sensitivity function in this case? **3 points**

Problem 3

In a SISO closed loop system with the loop gain $L(s) = G(s)F_y(s)$, the open loop system has a zero in $s = z > 0$. Assume that the amplitude $|L(i\omega)|$ is monotonously decreasing with frequency. Which is the theoretically highest possible cut-off frequency that can be used, if the phase margin must be at least 30^o?

Hint: Write the loop gain as

$$L(s) = \underbrace{\frac{z-s}{z+s}}_{L_1(s)} \tilde{L}(s)$$

What can be said about the amplitude and phase characteristics of $\tilde{L}(s)$?

6 points

Problem 4

Consider a simple feedback system where the nominal model is $G(s) = 1/s$, the feedback is a proportional regulator $F(s) = K$ and the true system is

$$G_o(s) = \frac{1}{s(1+sT)}$$

Both K and T can be assumed to be positive.

(a) Determine the relative model error $\Delta_G(s)$. **2 points**

(b) Assume that the criterion

$$\| \Delta_G \|_\infty \| T \|_\infty < 1$$

is used to examine for which values of K the closed loop system can be guaranteed to be stable. What is the result? **3 points**

(c) Assume that the criterion

$$\| \Delta_G T \|_\infty < 1$$

is used to examine for which values of K the closed loop system can be guaranteed to be stable. What is the result? **3 points**

(d) Determine the poles of the closed loop system. Find out when the closed loop system is asymptotically stable. **2 points**

Problem 5

Consider a double tank with inflows to both the upper (u_1) and the lower (u_2) tank. Assuming both tank levels can be measured, a local model around a working point can be written as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a & 0 \\ a & -a \end{pmatrix} x + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} u \\ y &= x \end{aligned}$$

where a and b are positive constants.

(a) Determine the transfer function of the system, and its poles and zeros. **2 points**

(b) Determine the singular values of the system. **4 points**

(c) Design a regulator based on Internal model control, using λ-tuning. Determine the sensitivity function of the closed loop system. **3 points**

(d) What are the singular values of the sensitivity function? **3 points**

Problem 6

Consider the scalar system

$$\begin{aligned} \dot{x} &= -x + u + v \\ y &= x + e \end{aligned}$$

where the process noise v and the measurement noise e both have constant intensities $\phi(\omega) \equiv 1$.

a) Assume that the state $x(t)$ is estimated using a standard observer

$$\dot{\hat{x}} = -\hat{x} + u + K(y - \hat{x})$$

with a constant gain K. Determine the stationary variance, say V, of the estimation error $\tilde{x} = x - \hat{x}$ as a function of K. **2 points**

(b) Determine what value of the observer gain that minimizes V. Let K^* denote this value of the gain. What is the minimum value of V? **4 points**

(c) What is the solution to the associated Riccati equation? **2 points**

(d) Assume next that the gain K^* is used, but that the observation process is improved by using a more accurate sensor, so that the measurement noise has intensity $\Phi_e(\omega) \equiv 1/3$. What is then the variance of the estimation error? **2 points**

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Control Design, December 21, 2006 — Answers and brief solutions

Problem 1

One has to solve the Riccati equation

$$0 = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S, \qquad L = Q_2^{-1}B^T S$$

If $L = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix}$, the loop gain $H(p)$ will be

$$H(p) = L(pI - A)^{-1}B = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \begin{pmatrix} p+1 & 0 \\ -1 & p \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\ell_1 p + \ell_2}{p(p+1)}$$

The Riccati equation becomes

$$\begin{aligned} 0 \;=\; & \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ & - \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

Written elementwise, this becomes

$$\begin{aligned} 0 &= -2s_{11} + 2s_{12} - s_{11}^2 \\ 0 &= -s_{12} + s_{22} - s_{11}s_{12} \\ 0 &= 0 + \alpha^2 - s_{12}^2 \end{aligned}$$

The last equation gives

$$s_{12} = \pm\alpha$$

The first equation then gives

$$s_{11}^2 + 2s_{11} \mp 2\alpha = 0 \Rightarrow s_{11} = -1 \pm [1 \pm 2\alpha]^{1/2}$$

In order to get s_{11} positive we must choose signs as

$$s_{12} = \alpha, \quad s_{11} = -1 + \sqrt{1 + 2\alpha}$$

The middle equation gives

$$s_{22} = s_{12}(1 + s_{11}) = \alpha\sqrt{1 + 2\alpha}$$

The feedback vector L is easily obtained as

$$L = \begin{pmatrix} s_{11} & s_{12} \end{pmatrix} = \begin{pmatrix} -1 + \sqrt{1 + 2\alpha} & \alpha \end{pmatrix}$$

The loop gain becomes

$$L(pI - A)^{-1}B = \frac{(-1+\sqrt{1+2\alpha})p + \alpha}{p(p+1)}$$

Problem 2

The general relation is $C(s) = [I + G(s)F(s)]^{-1}\,G(s)H(s)$ and hence

$$C(s) + G(s)F(s)C(s) = G(s)H(s)$$

In component form, this reads as

$$\begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix} + \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{pmatrix}\begin{pmatrix} C_{11} & 0 \\ 0 & C_{22} \end{pmatrix}$$
$$= \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{pmatrix}$$

which directly implies

$$C_{11} = \frac{G_{11}H_{11}}{1+G_{11}F_{11}}, \qquad C_{22} = \frac{G_{22}H_{22}}{1+G_{22}F_{22}}$$

and

$$(G_{21}F_{11} + G_{22}F_{21})\,C_{11} = G_{21}H_{11} + G_{22}H_{21}$$

In the general case the sensitivity function reads

$$S(s) = (I + G(s)F(s))^{-1} = \begin{pmatrix} 1+G_{11}F_{11} & 0 \\ G_{21}F_{11}+G_{22}F_{21} & 1+G_{22}F_{22} \end{pmatrix}^{-1} = \begin{pmatrix} S_{11} & 0 \\ S_{12} & S_{22} \end{pmatrix}$$

which implies

$$\begin{aligned} S_{11} &= (1+G_{11}F_{11})^{-1} \\ S_{22} &= (1+G_{22}F_{22})^{-1} \\ S_{12} &= -S_{11}(G_{21}F_{11}+G_{22}F_{21})S_{22} \end{aligned}$$

(a) When $F_{21} = 0$, it follows that

$$H_{21} = \frac{G_{21}F_{11}C_{11} - G_{21}H_{11}}{G_{22}} = \frac{G_{21}}{G_{22}}\,(F_{11}C_{11} - H_{11})$$

Furthermore, in this case

$$S_{12} = -S_{11}G_{21}F_{11}S_{22}$$

(b) When $H_{21} = 0$, it follows that

$$F_{21} = \frac{G_{21}H_{11} - G_{21}F_{11}C_{11}}{C_{11}G_{22}} = \frac{G_{21}}{C_{11}G_{22}}(H_{11} - F_{11}C_{11})$$

Furthermore, in this case

$$\begin{aligned} S_{12} &= -S_{11}\left(G_{21}F_{11} + G_{22}\frac{G_{21}}{C_{11}G_{22}}(H_{11}-F_{11}C_{11})\right)S_{22} \\ &= -S_{11}\frac{G_{21}}{C_{11}}\,(C_{11}F_{11} + H_{11} - F_{11}C_{11})\,S_{22} = -S_{11}S_{22}\frac{H_{11}}{C_{11}}G_{21} \\ &= -S_{11}S_{22}\frac{H_{11}G_{21}}{G_{11}H_{11}}(1+G_{11}F_{11}) = -S_{11}S_{22}\left(\frac{G_{21}}{G_{11}} + G_{21}F_{11}\right) \end{aligned}$$

Problem 3

The transfer function $\tilde{L}(s)$ has the same amplitude as $L(s)$. The amplitude $\tilde{L}(i\omega)$ is hence decreasing. It must therefore have a negative phase due to Bode's relations. As

$$\arg L(i\omega) = \arg L_1(i\omega) + \underbrace{\arg \tilde{L}(i\omega)}_{\leq 0} > -150^o$$

we get

$$\arg L_1(i\omega_c) > -150^o$$

which we rewrite as

$$\begin{aligned}
&\arg(z - i\omega_c) - \arg(z + i\omega_c) > -150^o \\
&-2\arctan(\omega_c/z) > -150^o \\
&\arctan(\omega_c/z) < 75^o \\
&\omega_c < z\tan(75^o) = (2+\sqrt{3})z \approx 3.73z
\end{aligned}$$

Problem 4

(a) As $G_o = G(1+\Delta_G)$ holds, we find that

$$\Delta_G(s) = \frac{G_o(s) - G(s)}{G(s)} = \frac{\frac{1}{s}\frac{1}{1+sT} - \frac{1}{s}}{\frac{1}{s}} = -\frac{sT}{1+sT}$$

(b) We find easily

$$\| \Delta_G \|_\infty = \sup_\omega |\Delta_G(i\omega)| = \sup_\omega \left|\frac{i\omega T}{1+i\omega T}\right| = \sup_\omega \frac{\omega T}{\sqrt{1+\omega^2 T^2}} = 1$$

Furthermore,

$$T(s) = \frac{G(s)F(s)}{1+G(s)F(s)} = \frac{K/s}{1+K/s} = \frac{K}{s+K} \Rightarrow \| T \|_\infty = 1$$

Hence, the stated sufficient stability condition is not satisfied for any value of K.

(c) In this case we need to examine

$$\| \Delta_G(s)T(s) \|_\infty = \| \frac{-sKT}{(s+K)(1+sT)} \|_\infty$$

Here we have

$$|\Delta_G(i\omega)T(i\omega)|^2 = \frac{\omega^2 K^2 T^2}{(K-\omega^2 T)^2 + \omega^2(1+KT)^2}$$

Seek maximum with respect to ω^2! This leeds to

$$\begin{aligned}
&K^2T^2\left[\omega^4 T^2 + \omega^2(1+K^2T^2) + K^2\right] - \omega^2 K^2 T^2\left[2\omega^2 T^2 + (1+K^2T^2)\right] = 0 \\
&\Rightarrow -K^2T^4\omega^4 + K^4T^2 = 0 \Rightarrow \omega^2 = K/T \\
&\| \Delta_G T \|_\infty^2 = \frac{K^3 T}{K/T(1+KT)^2} = \frac{K^2T^2}{(1+KT)^2} < 1
\end{aligned}$$

Hence, stability is guaranteed for all positive values of K.

(d) The closed loop system becomes

$$G_c(s) = \frac{G_o(s)K}{1 + G_o(s)K} = \frac{K}{s(1+sT)+K}$$

which apparently has both poles in the left half plan for all $K > 0$.

Problem 5

(a) The transfer function is

$$G(s) = b\begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix}^{-1} = \frac{b}{(s+a)^2}\begin{pmatrix} s+a & 0 \\ a & s+a \end{pmatrix}$$

The pole polynomial is easily found to be $(s+a)^2$. The system has a double pole in $s = -a$, and no zero.

(b) To determine the singular values of the system, we first determine the eigenvalues of the matrix

$$\begin{aligned} G(i\omega)G^*(i\omega) &= \frac{b^2}{[(i\omega+a)(-i\omega+a)]^2}\begin{pmatrix} i\omega+a & 0 \\ a & i\omega+a \end{pmatrix}\begin{pmatrix} -i\omega+a & a \\ 0 & -i\omega+a \end{pmatrix} \\ &= \frac{b^2}{(\omega^2+a^2)^2}\begin{pmatrix} a^2+\omega^2 & a^2+ai\omega \\ a^2-ai\omega & 2a^2+\omega^2 \end{pmatrix} = \frac{b^2}{(\omega^2+a^2)^2}Q(\omega) \end{aligned}$$

The characteristic polynomial of the matrix $Q(\omega)$ becomes

$$\det[\lambda - Q(\omega)] = \lambda^2 + \lambda(-3a^2 - 2\omega^2) + (a^4 + 2a^2\omega^2 + \omega^4)$$

The eigenvalues are thus

$$\begin{aligned} \lambda &= \frac{3a^2+2\omega^2}{2} \pm \left[\left(\frac{3a^2+2\omega^2}{2}\right)^2 - \left(a^4 + 2a^2\omega^2 + \omega^4\right)\right]^{1/2} \\ &= \frac{3a^2+2\omega^2}{2} \pm \frac{1}{2}\left[5a^4 + 4a^2\omega^2\right]^{1/2} \end{aligned}$$

The singular values of the system are the square roots of the eigenvalues to $G(i\omega)G^*(i\omega)$, which gives the two singular values

$$\sigma(\omega) = \frac{b}{\sqrt{2}(a^2+\omega^2)}\sqrt{3a^2 + 2\omega^2 \pm \sqrt{4a^2\omega^2 + 5a^4}}$$

(c) Set the Q factor to

$$Q(s) = \frac{1}{\lambda s + 1}G^{-1}(s) = \frac{1}{\lambda s + 1}\begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix}\frac{1}{b}$$

The regulator becomes

$$\begin{aligned} F_y(s) &= [I - Q(s)G(s)]^{-1}Q(s) \\ &= \frac{1}{\lambda s}G^{-1}(s) = \frac{1}{\lambda s}\begin{pmatrix} s+a & 0 \\ -a & s+a \end{pmatrix}\frac{1}{b} \end{aligned}$$

The sensitivity function will be

$$S(s) = I - G(s)Q(s) = \frac{\lambda s}{\lambda s + 1}I$$

(d) $S(s)$ has two singular values, which are identical:

$$\sigma(\omega) = \left|\frac{\lambda i\omega}{\lambda i\omega + 1}\right| = \frac{\lambda\omega}{\sqrt{\lambda^2\omega^2 + 1}}$$

Problem 6

To treat the general case, let the measurement noise have intensity r_2. The estimation error $\tilde{x}$ satisfies

$$\dot{\tilde{x}} = (-1 - K)\tilde{x} - Ke + v$$

Applying the Lyapunov equation then gives easily

$$V(K) = \frac{K^2 r_2 + 1}{2(1 + K)}$$

(a) Setting $r_2 = 1$ gives

$$V(K) = \frac{K^2 + 1}{2(1 + K)}$$

(b) The minimizing element of $V(K)$ is found to be $K^* = -1 + \sqrt{2}$. Further, the minimal value turns out to be $V = \sqrt{2} - 1$.

(c) The associated Riccati equation is

$$0 = -P - P + 1 - P^2 \times 1^2/r_2$$

which leads to

$$P^2 + 2r_2 P - r_2 = 0$$

with the solution $P = -r_2 \pm \sqrt{r_2^2 + r_2}$. As $r_2 = 1$ in part (c), the positive solution is $P = \sqrt{2} - 1$, as V in part (b).

(d) The variance using the fixed observer gain K^* for the noise intensity $r_2 = 1/3$ becomes

$$V = \frac{(K^*)^2/3 + 1}{2(1 + K^*)} = \frac{\sqrt{2}}{2} - \frac{1}{3} \approx 0.374$$