Uppsala University
Department of Information Technology
Systems and Control
Professor Torsten Söderström

Final exam: Control Design (Reglerteknisk design, 1TT492)

Date: December 19, 2008

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English.

Problem 6 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators. Note that the following are **not allowed**: Exempelsamling med lösningar, copies of OH transparencies.

Good luck!

Problem 1

Consider an oscillative system with transfer function

$$G(s) = \frac{\omega_o^2}{s^2 + 2\zeta\omega_o s + \omega_o^2}$$

Design a feedback regulator using Internal Model Control and λ tuning. Will the regulator be integrating? **5 points**

Problem 2

Consider a two-input, two-output system with the transfer function

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & \frac{10}{s+1} \\ \frac{1}{s+5} & \frac{5}{s+3} \end{pmatrix}$$

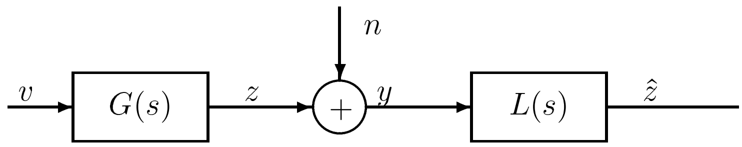
(a) Determine RGA($G(0)$). **4 points**

(b) Which input-output pairing should be preferred? **2 points**

Problem 3

Civ intends to use Kalman filter theory as a methodology to filter a noisy measurement signal. Consider the setup displayed in the figure below. Here $z(t)$ is the useful signal, $n(t)$ the measurement noise and $y(t)$ the noisy measurements. The filter to be designed is denoted L, which output $\hat{z}(t)$ should be a good estimate of the useful signal $z(t)$. It is desired that $L(s)$ should be a low-pass filter with the characteristics

$$\begin{aligned} L(0) &= 1 \\ L(s) & \quad \text{decreasing at least as } 1/s \text{ for large } s. \end{aligned}$$



(a) Civ first tried to model the signal $z(t)$ as a first order filtered signal with the transfer function

$$G(s) = \frac{b}{s+a}$$

and with a white input signal $v(t)$ having intensity r_v. The measurement noise is also assumed to be white noise, and to have intensity r_e.

Show that the setup can in state space form be written as

$$\begin{aligned} \dot{x} &= -ax + \beta v \\ y &= (b/\beta)x + n \end{aligned}$$

What values can the parameter β take? Writing the general observer in the form

$$\dot{\hat{x}} = -a\hat{x} + k\left[y - (b/\beta)\hat{x}\right]$$

derive the corresponding filter transfer function $L(s)$. **2 points**

(b) What constraints on the model $G(s)$ are needed to meet the specified properties of the filter $L(s)$? **2 points**

(c) Using the constraints on $G(s)$ derived in part (b), derive the Kalman filter. How is it influenced by the choice of the parameter β? **3 points**

(d) Show that the way the noise intensities r_v and r_e influence in the Kalman filter is only through the ratio r_v/r_e, and write out this dependence explicitly. **2 points**

(e) Assume instead that the signal model is taken as the second order system

$$G(s) = \frac{b}{s(s+a)}, \quad (a > 0, b > 0)$$

Will the constraint $L(0) = 1$ be satisfied? **3 points**

Problem 4

Consider H_2 control of the system

$$G(s) = \frac{1}{s+1}$$

with the weightings

$$W_S(s) = \frac{\alpha}{s}\ (\alpha > 0),\ W_T(s) = 1,\ W_u(s) = \gamma$$

(a) Write the extended model in state-space form. **4 points**

(b) Find the solution of the associated Riccati equation. **5 points**

(c) Determine the optimal H_2 regulator. **2 points**

Problem 5

Consider a system with the transfer function

$$G(s) = \frac{1}{10s+1}\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

where ε is a real-valued number.

(a) When designing a controller for the system, Civerth chose not to consider the crosscouplings in the system, that is he used the simplified model

$$G_s(s) = \frac{1}{10s+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Determine the relative model uncertainty Δ_G. **2 points**

(b) Civerth made a choice to design two decoupling PI-regulators, which leads to the feedback

$$F_y(s) = \begin{pmatrix} K_1\frac{T_1s+1}{T_1s} & 0 \\ 0 & K_2\frac{T_2s+1}{T_2s} \end{pmatrix}$$

Determine the regulator parameters, based on the simplified open loop model, so that the closed loop system becomes $G_c(s) = \frac{1}{s+1}I$. **2 points**

(c) Use some suitable robustness criterion to decide for which values of ε one can guarantee that the closed loop system is stable, when the feedback in (b) is applied. **3 points**

(d) Determine for precisely what values of ε the closed loop system is indeed stable. **3 points**

Problem 6

Consider feedback control of a servo system with the dynamics

$$G(s) = \frac{1}{s(s+1)}$$

The feedback is assumed to have the structure

$$u = r - \frac{b_0p + b_1}{p+a}y$$

where r is a reference signal, and p denotes the differentiation operator.

(a) Determine the regulator parameters $a,\ b_0,\ b_1$ so that the closed loop system has its poles in $s=-1, s=-1+i, s=-1-i$. **2 points**

(b) Determine the complementary sensitivity function. **2 points**

(c) Assume that the open loop system does contain some unmodelled dynamics in terms of a weak resonance, so that the true transfer function is

$$\begin{aligned} G_p(s) &= G(s)G_1(s) \\ G_1(s) &= \frac{\omega_o^2}{s^2+2\zeta\omega_0 s+\omega_0^2} \end{aligned}$$

where the relative damping ζ is a small positive number.

Represent the deviation of $G_p(s)$ from $G(s)$ in terms of a relative error $\Delta_G(s)$. **2 points**

Uppsala University
Department of Information Technology
Systems and Control
Prof Torsten Söderström

Control Design, December 19, 2008 — Answers and brief solutions

Problem 1

The transfer function $Q(s)$ becomes

$$Q(s) = \frac{1}{(\lambda s+1)^2} G^{-1}(s) = \frac{1}{(\lambda s+1)^2} \frac{s^2 + 2\zeta\omega_o s + \omega_o^2}{\omega_o^2}$$

and the regulator will be

$$\begin{aligned} F_y(s) &= [1 - Q(s)G(s)]^{-1} Q(s) \\ &= \frac{1}{1 - \frac{1}{(\lambda s+1)^2}} \frac{s^2 + 2\zeta\omega_o s + \omega_o^2}{(\lambda s+1)^2 \omega_o^2} \\ &= \frac{1}{\lambda^2 s^2 + 2\lambda s} \frac{s^2 + 2\zeta\omega_o s + \omega_o^2}{\omega_o^2} \end{aligned}$$

As $F_y(0) = \infty$, the regulator is integrating.

Problem 2

(a) In this case

$$G(0) = \begin{pmatrix} 1/2 & 10 \\ 1/5 & 5/3 \end{pmatrix}.$$

Hence

$$\begin{aligned} \text{RGA}(G(0)) &= \begin{pmatrix} 1/2 & 10 \\ 1/5 & 5/3 \end{pmatrix} \odot \frac{1}{5/6 - 2} \begin{pmatrix} 5/3 & -1/5 \\ -10 & 1/2 \end{pmatrix} \\ &= \frac{-6}{7} \begin{pmatrix} 5/6 & -2 \\ -2 & 5/6 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -5 & 12 \\ 12 & -5 \end{pmatrix} \end{aligned}$$

(b) Avoid pairing leading to negative diagonal elements in RGA. Hence combine u_1 with y_2, and u_2 with y_1.

Problem 3

(a) The state-space model means that the transfer function from v to z is

$$C(sI - A)^{-1} N = (b/\beta) \frac{1}{s+a} \beta = G(s)$$

Any non-zero value of β can be used.

The observer leads to

$$\begin{aligned}\hat{z} &= (b/\beta)\hat{x} \\ &= \frac{(b/\beta)k}{p + a + kb/\beta} y\end{aligned}$$

so

$$L(s) = \frac{bk/\beta}{s + a + bk/\beta}$$

(b) The condition $L(0) = 1$ leads to that Civ must make the choice

$$a = 0$$

(c,d) The Riccati equation

$$0 = AP + PA^T + NR_v N^T - PC^T R_e^{-1} CP$$

becomes in this case simply

$$0 = 0 + 0 + \beta^2 r_v - p^2 (b/\beta)^2 / r_e$$

with the solution

$$p = \frac{\beta^2}{b} \sqrt{r_v r_e}$$

The corresponding filter gain is

$$k = (pb)/(\beta r_e) = \beta \sqrt{\frac{r_v}{r_e}}$$

In the filter the parameter

$$bk/\beta = b\sqrt{\frac{r_v}{r_e}}$$

does not depend at all on β.

(e) Represent the system as

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -a & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ b \end{pmatrix} v \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + e\end{aligned}$$

Then it holds

$$\begin{aligned}L &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p + a + k_1 & -1 \\ k_2 & p \end{pmatrix}^{-1} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= \frac{1}{p(p + a + k_1) + k_2} \begin{pmatrix} p & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= \frac{pk_1 + k_2}{p^2 + (a + k_1)p + k_2}\end{aligned}$$

which shows in particular that $L(0) = 1$, no matter of the values of k_1 and k_2.

Problem 4

(a) Write the system to be controlled as

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ y &= x_1 \end{aligned}$$

Set

$$\begin{aligned} z_1 &= \gamma u \\ z_2 &= x_1 \\ z_3 &= \frac{\alpha}{p}(x_1 + w) \end{aligned}$$

Choose the second state as

$$x_2 = z_3$$

This gives together the extended model

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 \\ \alpha & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ \alpha \end{pmatrix} w \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + w \\ z &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix} u \end{aligned}$$

Check the matrix

$$A - NC = \begin{pmatrix} -1 & 0 \\ \alpha & 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

As it does not have any eigenvalue in the right half plane, the model is in innovations form.

(b) The Riccati equation

$$0 = A^T S + SA - M^T M - SB(D^T D)^{-1} B^T S$$

gives in partitioned form with

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} + \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \alpha & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &- \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\gamma^2} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \end{aligned}$$

Evaluating the matrix equation componentwize,

$$
\begin{aligned}
0 &= -2s_{11} + 2\alpha s_{12} + 1 - s_{11}^2/\gamma^2 \\
0 &= -s_{12} + \alpha s_{22} - s_{11}s_{12}/\gamma^2 \\
0 &= 1 - s_{12}^2/\gamma^2
\end{aligned}
$$

The second equation gives

$$s_{22} = \frac{s_{12}}{\alpha}\left(1 + \frac{s_{11}}{\gamma^2}\right)$$

from which we conclude that $s_{12} > 0$. Hence, the third equation gives

$$s_{12} = \gamma$$

The first equation implies

$$
\begin{aligned}
0 &= s_{11}^2 + 2\gamma^2 s_{11} - 2\alpha\gamma^3 - \gamma^2 \\
s_{11} &= -\gamma^2 \pm \left[\gamma^4 + 2\alpha\gamma^3 + \gamma^2\right]^{1/2}
\end{aligned}
$$

The positive sign must be chosen, as $s_{11} > 0$ must hold. Hence,

$$S = \begin{pmatrix} -\gamma^2 + (\gamma^4 + 2\alpha\gamma^3 + \gamma^2)^{1/2} & \gamma \\ \gamma & \frac{\gamma}{\alpha}\left(1 + \frac{2\alpha}{\gamma} + \frac{1}{\gamma^2}\right)^{1/2} \end{pmatrix}$$

(c) For the optimal regulator it holds

$$L = (D^T D)^{-1} B^T S = \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} = \frac{1}{\gamma^2}\begin{pmatrix} s_{11} & s_{12} \end{pmatrix}$$

and then

$$
\begin{aligned}
F_y(s) &= L\left[sI - A + B(D^T D)^{-1} B^T S + NC\right]^{-1} N \\
&= \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \begin{pmatrix} s + 1 + \ell_1 & \ell_2 \\ -\alpha + \alpha & s \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \\
&= \frac{1}{s(s + 1 + \ell_1)} \begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} \begin{pmatrix} s & -\ell_2 \\ 0 & s + 1 + \ell_1 \end{pmatrix} \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \\
&= \frac{\alpha \ell_2 (s+1)}{s(s + 1 + \ell_1)} = \frac{\alpha \frac{s_{12}}{\gamma^2}(s+1)}{s(s + 1 + \frac{s_{11}}{\gamma^2})} \\
&= \frac{\alpha\gamma(s+1)}{s\left(\gamma^2 s + \gamma^2 - \gamma^2 + \sqrt{\gamma^4 + 2\alpha\gamma^3 + \gamma^2}\right)} \\
&= \frac{\alpha(s+1)}{s\left(\gamma s + \sqrt{\gamma^2 + 2\alpha\gamma + 1}\right)}
\end{aligned}
$$

Problem 5

(a) Introduce $G(s) = [I + \Delta_G(s)]G_s(s)$, which gives

$$\Delta_G(s) = G(s)G_s^{-1}(s) - I = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

(b) The closed loop system will be

$$G_c(s) = \frac{K(1+sT)}{Ts(10s+1) + K(1+sT)} I = \frac{1}{s+1} I$$

precisely when $T_1 = T_2 = 10,\ K_1 = K_2 = 10$.

(c) The robustness criteria for the uncertainty in part (a) is $\| T\Delta_G \|_\infty < 1$, that is the largest singular value of $T(i\omega)\Delta_G(i\omega)$ must be less than one for all frequencies ω. Here $T(s) = \frac{1}{s+1} I$. Using $\Delta_G(s)$ as in part (b),

$$T(s)\Delta_G(s) = \frac{1}{s+1} \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

The singular values of this matrix happens to be equal, and

$$\sigma_1(\omega) = \sigma_2(\omega) = \frac{|\varepsilon|}{\sqrt{\omega^2 + 1}}$$

Hence, robust stability is guaranteed if

$$|\varepsilon| < \sqrt{\omega^2 + 1} \quad \forall \omega$$

which finally leads to the condition

$$|\varepsilon| < 1$$

(d) The loop gain will be

$$L(s) = G(s)F_y(s) = \frac{1}{s} \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

The sensitivity function becomes

$$S(s) = [I + L(s)]^{-1} = \frac{s}{(s+1)^2 - \varepsilon^2} \begin{pmatrix} s+1 & -\varepsilon \\ -\varepsilon & s+1 \end{pmatrix}$$

One can then see that the system is stable precisely when $(s+1)^2 - \varepsilon^2$ has all zeros strictly in the left half plane, which is the case when $|\varepsilon| < 1$.

Problem 6

(a) The closed loop characteristic equation can be written as

$$1 + \frac{1}{s(s+1)} \frac{b_0 s + b_1}{s+a} = 0$$
$$\Rightarrow s(s+1)(s+a) + (b_0 s + b_1) = 0$$
$$\Rightarrow s^3 + s^2(a+1) + s(a+b_0) + b_1 = 0$$

Compare this to the specified characteristic polynomial which is

$$(s+1)(s+1+i)(s+1-i) = s^3 + 3s^2 + 4s + 2$$

and comparing coefficients leads to

$$a = 2, \quad b_0 = 2, \quad b_1 = 2$$

(b) The loop gain is

$$L(s) = F(s)G(s) = \frac{2(s+1)}{s+2} \frac{1}{s(s+1)} = \frac{2}{s(s+2)}$$

Hence the complementary sensitivity function will be

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{2}{s^2+2s+2}$$

(c) Utilizing the definition

$$G_p(s) = G(s)\left(1 + \Delta_G(s)\right)$$

gives easily

$$\Delta_G(s) = G_1(s) - 1 = -\frac{s^2 + 2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$