Uppsala University
Department of Information Technology
Systems and Control
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Final exam: Control Design (Reglerteknisk design, 1TT492)

Date: December 14, 2009

Responsible examiner: Torsten Söderström

Preliminary grades: 3 = 23–32p, 4 = 33–42p, 5 = 43–50p.

Instructions

The solutions to the problems can be given in Swedish or in English, except for Problems 1 and 2 that *must* be solved in English.

Problem 1 is an alternative to the homework assignment. (In case you choose to hand in a solution to Problem 1 you will be accounted for the best performance of the homework assignments and Problem 1.)

Solve each problem on a separate page.

Write your name on every page.

Provide motivations for your solutions. Vague or lacking motivations may lead to a reduced number of points.

Aiding material: Textbooks in automatic control (such as 'Reglerteori – flervariabla och olinjära metoder', 'Reglerteknik – Grundläggande teori', and others), mathematical handbooks, collection of formulas (formelsamlingar), textbooks in mathematics, calculators, copies of OH transparencies. Note that the following is **not allowed**: Exempelsamling med lösningar.

Good luck!

Problem 1

(a) Consider the system

$$G(s) = \frac{K(1-s)}{(s+1)^2}$$

What is the utmost highest possible cut-off frequency that can be chosen when controlling this system with a feedback $u(t) = -y(t)$? **3 points**

(b) How high can the gain K be chosen, if the system is controlled with unit feedback ($u = -y$) before losing stability? **3 points**

Problem 2

Consider an undamped oscillator given by the state space model

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

It is to be controlled by state feedback of the form

$$u = -\begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} x + mr$$

where r is an external reference signal.

(a) How should m be designed, as a function of ℓ_1 and ℓ_2 if the output y should in stationarity follow the reference signal r without any static error? **2 points**

(b) Assume that r is changed by a unit step, and that the maximal value of the input u occurs just as the step starts. Further, assume that the input must be bounded so that

$$|u(t)| \leq u_o, \qquad \text{all } t$$

What does this condition imply for ℓ_1 and ℓ_2? **2 points**

(c) Assume that the feedback gains ℓ_1 and ℓ_2 are determined from an LQ problem with the weighting matrices

$$Q_1 = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_2 = 1$$

What does the condition treated in part (b) imply for the selection of the weight ρ? **4 points**

Problem 3

Assuming no disturbances are present, the following model is assumed to describe a dynamic system.

$$\frac{d}{dt}y(t) + y(t) = u(t).$$

The system should be controlled by a $\mathcal{H}_2$ designed regulator, where the following weights apply:

$$W_S(s) = \frac{1}{s}, \quad W_T(s) = 1 \quad \text{och} \quad W_u = 1.$$

(a) Determine a state space model for the extended system in the standard form

$$\begin{aligned}
\frac{d}{dt}x(t) &= Ax(t) + Bu(t) + Nw(t) \\
z(t) &= Mx(t) + Du(t) \\
y(t) &= Cx(t) + w(t)
\end{aligned}$$

Check if the required conditions on A, N, C, M and D are satisfied.

4 points

(b) Determine the transfer function $F_y(p)$, where the resulting regulator is $u(t) = -F_y(p)y(t)$ **4 points**

Problem 4

Consider control design using the Internal Model Control principle with the so called λ-tuning for the system with a transfer function

$$G(s) = \frac{1-s}{s(s+1)}$$

Of the possible options, let the nonminimum phase zero be treated by neglecting it when taking $G^{-1}(s)$ in the design.

(a) Determine the transfer function of the regulator. **2 points**

(b) Detemine the closed-loop system and its poles. **2 points**

(c) Assume that there is a process noise disturbance acting as a ramp, $w(t) = t$. What is the stationary value of the output signal in that case? **4 points**

Problem 5

Consider determining an unknown velocity from discrete-time measurements of the position using a Kalman filter. The Kalman filter is based on the model

$$\begin{aligned}
x(kh+h) &= \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(kh) + \begin{pmatrix} 0 \\ h \end{pmatrix} w(kh) \\
y(kh) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(kh)
\end{aligned}$$

The first state variable, $x_1(kh)$ denotes the measurement of the position at time $t = kh$, and $x_2(kh)$ is the velocity. In the model the acceleration is $w(kh)$ and it is modelled as a white noise process. There is no measurement noise in the model.

a) Determine the Kalman filter for the model. What is the value of the Kalman gain vector K in this case? What is the solution to the associated Riccati equation? **4 points**

b) Use the Kalman filter to determine how the estimated velocity $\hat{x}_2(kh)$ depends on the measurements of the position. Interprete the result. How large is the variance of the estimation error $x_2(kh) - \hat{x}_2(kh)$? **3 points**

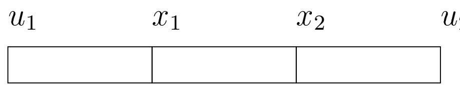
c) Suppose that the underlying movement is perfectly linear, so that the observations are given by

$$y(kh) = \alpha_0 + \alpha_1 k$$

with some unknown parameters α_0 and α_1. Use the derived Kalman filter to determine the estimate $\hat{x}_1(kh)$. Compare with the observation $y(kh)$. **2 points**

Problem 6

We want to control the temperature in a long copper rod by heating or cooling its endpoints. Principally, this problem is described by a partial differential equation. To simplify the problem we assume that the temperature profile in the rod can be approximated by the temperatures x_1 and x_2 at two points. The temperatures in the end points are the inputs, u_1 and u_2. All temperatures are relative to the temperature of the surroundings.



We get the following equations

$$\begin{aligned} \dot{x}_1 &= \alpha(u_1 - x_1) + \alpha(x_2 - x_1) \\ \dot{x}_2 &= \alpha(x_1 - x_2) + \alpha(u_2 - x_2) \end{aligned}$$

where α is a constant that depends on the coefficient of thermal conductivity and the specific heat of the rod. For simplicity, let $\alpha = 1$. Consider the problem of controlling the temperature in x_1 and x_2 with $u = u_1$ only, assuming $u_2 = 0$.

(a) Is the system controllable? **1 point**

(b) Assume that the input $u(t)$ is a constant step. What is then the steady-state value of the state vector? **2 points**

(c) Is it possible to find an input signal and a final time τ such that for $u(t), 0 \leq t \leq \tau$ it holds $x(\tau) = x^*$ for any arbitrary vector x^*? **2 points**

(d) Is it possible to find an input signal and a time τ such that $u(t)$, such that $x(\tau + t) = x^*$ for all $t \geq 0$ and for any arbitrary vector x^*? **3 points**

(e) Write the system dynamics as

$$\dot{x} = Ax + Bu$$

and define the matrix $W(\tau)$ as

$$W(\tau) = \int_0^\tau e^{A(\tau-s)} B B^T e^{A^T(\tau-s)} ds$$

Consider the specific input signal

$$u(t) = B^T e^{A^T(\tau-t)} W^{-1}(\tau) \left(x^* - e^{A\tau} x(0) \right)$$

where $x(0)$ is the initial state vector. Determine the state vector at time $t = \tau$. **3 points**

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Control design, December 14, 2009 — Answers and brief solutions

Problem 1

(a) The highest possible cut-off frequency is achieved when there is no phase margin at all. Determine ω_c from $\arg G(i\omega_c) = -\pi$. This leads to

$$-\pi = \arg(1 - i\omega_c) - 2\arg(1 + i\omega_c) \quad = \quad -\arctan(\omega_c) - 2\arctan(\omega_c)$$

and hence

$$3\arctan(\omega_c) = \pi \Rightarrow \arctan(\omega_c) = \pi/3 \Rightarrow \omega_c = \sqrt{3}$$

(b) Determine K from

$$|G(i\omega_c)| \leq 1 \Rightarrow \frac{K}{\sqrt{\omega_c^2 + 1}} \leq 1 \Rightarrow K \leq 2$$

Problem 2

(a) A static gain equal to one from r to y gives

$$\begin{aligned} m &= -\frac{1}{C(A - BL)^{-1}B} \\ &= \frac{-1}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 - \ell_1 & -\ell_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \\ &= \frac{-(1 + \ell_1)}{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\ell_2 & -1 \\ 1 + \ell_1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = (1 + \ell_1) \end{aligned}$$

(b) One gets directly from (a)

$$|1 + \ell_1| \leq u_o$$

(c) Denote the solution to the Riccati equation

$$0 = A^T S + SA + Q_1 - SBQ_2^{-1}B^T S$$

by

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$$

Then the feedback vector L is

$$\begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} = Q_2^{-1} B^T S = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} = \begin{pmatrix} s_{12} & s_{22} \end{pmatrix}$$

Hence, the design constraint becomes

$$|1 + s_{12}| \leq u_o$$

Evaluating both sides of the Riccati equation leads to the nonlinear system of equations

$$\begin{aligned} 0 &= -2s_{12} + \rho - s_{12}^2 \\ 0 &= -s_{22} + s_{11} - s_{12}s_{22} \\ 0 &= 2s_{12} - s_{22}^2 \end{aligned}$$

Here the first equation gives s_{12}, then the third gives s_{22}, and finally s_{11} can be found from the second one. For this problem, though, it is sufficient to solve the first equation to get

$$s_{12} = -1 \pm \sqrt{1 + \rho}$$

From the third equation we find that s_{12} must be positive. Hence we get $s_{12} = -1 + \sqrt{1 + \rho}$ and

$$|m| \leq u_o \Rightarrow |1 + s_{12}| \leq u_o \Rightarrow \sqrt{1 + \rho} \leq u_o \Rightarrow \rho \leq u_o^2 - 1$$

Problem 3

(a) If x_1 is the unperturbed output and $x_2 = W_S(Gu + w)$, the following model is obtained:

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w \\ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + w \end{aligned}$$

Check the conditons:

$$\det(\lambda I - A + NC) = \det\left(\begin{pmatrix} \lambda + 1 & 0 \\ 0 & \lambda \end{pmatrix}\right) = \lambda(\lambda + 1) = 0$$

$\Longrightarrow A - NC$ has no eigenvalues in RHP.

Further,

$$D^T D = 1, D^T M = 0$$

Hence all conditions for the standard scheme are satisfied.

(b)

$$A^T S + SA + M^T M - SBB^T S = 0 \Longrightarrow \begin{cases} 2(s_{12} - s_{11}) + 1 - s_{11}^2 = 0 \\ s_{22} - s_{12} - s_{11}s_{12} = 0 \\ 1 - s_{12}^2 = 0 \end{cases}$$

$$\Longrightarrow S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \Longrightarrow L = \begin{pmatrix} 1 & 0 \end{pmatrix} S = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$F_y(p) = L\left(pI - A + BB^T S + NC\right)^{-1} N = \frac{p+1}{p(p+2)}$$

Problem 4

(a) We have in this case

$$Q(s) = \frac{s(s+1)}{(1+\lambda s)^2}$$

which leads to

$$\begin{aligned} F_y(s) &= \frac{Q(s)}{1 - Q(s)G(s)} = \frac{s(s+1)}{(1+\lambda s)^2} \frac{1}{1 - \frac{1-s}{(1+\lambda s)^2}} \\ &= \frac{s(s+1)}{(2\lambda+1)s + \lambda^2 s^2} = \frac{(s+1)}{(2\lambda+1) + \lambda^2 s} \end{aligned}$$

(b) The closed loop system becomes

$$G_c(s) = Q(s)G(s) = \frac{1-s}{(1+\lambda s)^2}$$

and it has two poles in $s = -1/\lambda$.

(c) The sensitivity function is

$$S(s) = 1 - Q(s)G(s) = \frac{(1+\lambda s)^2 - (1-s)}{(1+\lambda s)^2} = \frac{(2\lambda+1)s + \lambda^2 s^2}{(1+\lambda s)^2}$$

As $S(0) = 0$, a constant disturbance does not lead to any stationary error. A ramp disturbance leads to the stationary output

$$\lim_{t\to\infty} y(t) = \frac{dS}{ds}(0) = 2\lambda + 1$$

Problem 5

a) The Riccati equation

$$P = APA^T + NR_1N^T - APC^T\left(CPC^T + R_2\right)^{-1} CPA^T$$

becomes in this case, after setting

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} &= \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & rh^2 \end{pmatrix} \\ &- \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \end{pmatrix} \frac{1}{p_{11}+0} \begin{pmatrix} p_{11} & p_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix} \end{aligned}$$

Evaluating the different elements leads to the system of equations

$$\begin{aligned} p_{11} &= p_{11} + h^2 p_{22} + 2hp_{12} - (p_{11} + hp_{12})^2 / p_{11} \\ p_{12} &= p_{12} + hp_{22} - (p_{11} + hp_{12}) p_{12}/p_{11} \\ p_{22} &= p_{22} + h^2 r - p_{12}^2/p_{11} \end{aligned}$$

leading to

$$\begin{aligned} 0 &= h^2(p_{11}p_{22} - p_{12}^2) - p_{11}^2 && (0.1) \\ 0 &= h(p_{11}p_{22} - p_{12}^2) - p_{11}p_{12} && (0.2) \\ 0 &= h^2 r p_{11} - p_{12}^2 && (0.3) \end{aligned}$$

Comparing (0.1) and (0.2) leads to

$$p_{11}^2 = hp_{11}p_{12}$$

and

$$p_{12} = p_{11}/h$$

Inserting this result into (0.3) gives

$$0 = h^2 r p_{11} - p_{11}^2/h^2$$

leading to

$$p_{11} = h^4 r, \quad p_{12} = h^3 r$$

Equation (0.2) then finally gives

$$h^5 r p_{22} - h^7 r^2 - h^7 r^2 = 0$$

and

$$p_{22} = 2h^2 r$$

The total P matrix is hence

$$P = r \begin{pmatrix} h^4 & h^3 \\ h^3 & 2h^2 \end{pmatrix}$$

The Kalman gain K becomes

$$K = \begin{pmatrix} p_{11} + hp_{12} \\ p_{12} \end{pmatrix} \frac{1}{p_{11}} = \begin{pmatrix} 1 + hp_{12}/p_{11} \\ p_{12}/p_{11} \end{pmatrix} = \begin{pmatrix} 2 \\ 1/h \end{pmatrix}$$

(b) The filter output $\hat{x}(kh)$ can be written as

$$
\begin{aligned}
\hat{x}(kh) &= (qI - A + KC)^{-1} Ky(kh) \\
&= \begin{pmatrix} q-1+2 & -h \\ 1/h & q-1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1/h \end{pmatrix} y(kh) \\
&= \frac{1}{q^2} \begin{pmatrix} q-1 & h \\ -1/h & q+1 \end{pmatrix} \begin{pmatrix} 2 \\ 1/h \end{pmatrix} y(kh) = \frac{1}{q^2} \begin{pmatrix} 2q-1 \\ \frac{1}{h}(q-1) \end{pmatrix} y(kh)
\end{aligned}
$$

This leads in particular to

$$
\hat{x}_2(kh) = \frac{1}{q^2}\frac{1}{h}(q-1)y(kh) = \frac{1}{h}\left[y(kh-h) - y(kh-2h)\right]
$$

The estimate of the velocity is hence equal to normed difference of the measured position (which makes perfect sense). The variance of the estimation error is $p_{22} = 2h^2 r$.

(c) The Kalman filter provides the estimate

$$
\hat{x}_1(kh) = \frac{1}{q^2}(2q-1)\,y(kh) = 2y(kh-h) - y(kh-2h)
$$

For the given data model, we therefore get

$$
\begin{aligned}
\hat{x}_1(kh) &= 2y(kh-h) - y(kh-2h) = 2\left(\alpha_0 + \alpha_1(kh-h)\right) - \left(\alpha_0 + \alpha_1(kh-2h)\right) \\
&= 2\alpha_0 + 2\alpha_1 kh - 2\alpha_1 h - \alpha_0 + \alpha_1 kh + 2\alpha_1 h = \alpha_0 + \alpha_1 kh \\
&= y(kh)
\end{aligned}
$$

Problem 6

(a) The system can be written as

$$
\dot{x} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u
$$

The controllability matrix therefore becomes

$$
W_c = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}
$$

which is nonsingular. The system is hence controllable.

(b) For a fixed input u, the steady-state value of the state vector x satisfies

$$
\begin{aligned}
0 &= Ax + Bu \\
x &= -A^{-1}Bu = -\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\
&= -\frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} u
\end{aligned}
$$

(c) Yes, as the system is controllable.

(d) No, as according to (b) any steady-state value of the state vector is proportional to $(2\ 1)^T$.

(e) The standard equation for solution of the state equation gives

$$\begin{aligned} x(\tau) &= e^{A\tau}x(0) + \int_0^\tau e^{A(\tau-s)}Bu(s)ds \\ &= e^{A\tau}x(0) + \int_0^\tau e^{A(\tau-s)}BB^T e^{A^T(\tau-s)}W^{-1}(\tau)\left(x^* - e^{A\tau}x(0)\right)ds \\ &= e^{A\tau}x(0) + \int_0^\tau e^{A(\tau-s)}BB^T e^{A^T(\tau-s)}ds\ W^{-1}(\tau)\left(x^* - e^{A\tau}x(0)\right) \\ &= e^{A\tau}x(0) + W(\tau)W^{-1}(\tau)\left(x^* - e^{A\tau}x(0)\right) = x^* \end{aligned}$$