Chapter 1

2009-01-26 AM

Exe 1.6.1

Let $I = (x_0, x_1)$ and $\lambda_0 = \frac{x_1 - x}{x_1 - x_0}$, $\lambda_1 = \frac{x - x_0}{x_1 - x_0}$. Verify

a) $\lambda_0 + \lambda_1 = 1$

b) $x_0\lambda_0 + x_1\lambda_1 = x$

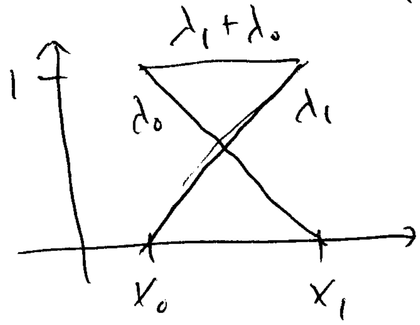
c) Draw $\lambda_0, \lambda_1, \lambda_0 + \lambda_1, x_0\lambda_0 + x_1\lambda_1, x_0\lambda_0, x_1\lambda_1$

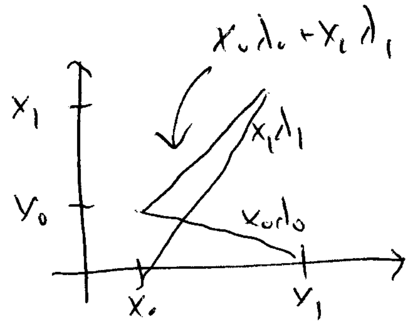
Sol a) $\lambda_0 + \lambda_1 = \frac{x_1 - \cancel{x} + \cancel{x} - x_0}{x_1 - x_0} = 1$

b) $x_0\lambda_0 + x_1\lambda_1 = x_0 \frac{x_1 - x}{x_1 - x_0} + x_1 \frac{x - x_0}{x_1 - x_0} = \frac{\cancel{x_0 x_1} - x_0 x + x_1 x - \cancel{x_0 x_1}}{x_0 - x_1}$

$= x \cdot \frac{x_1 - x_0}{x_1 - x_0} = x$

c)





Exe 1.6.2

Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, $x_1 = 1/6$, $x_2 = \frac{1}{2}$ and let V_h be the space of piecewise linear fcns on the partition.

a) Draw φ_1

b) Draw $v(x) = -\varphi_0 + \varphi_2 + 2\varphi_3$ and its derivative v_x

c) Draw the mesh fcn $h(x)$

d) What is the dimension of V_h?

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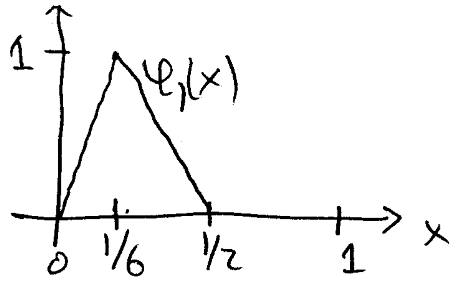
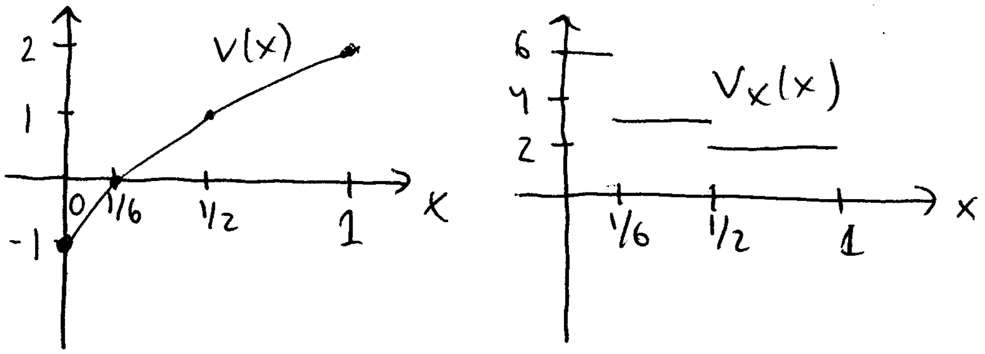
Sol (1.6.2)

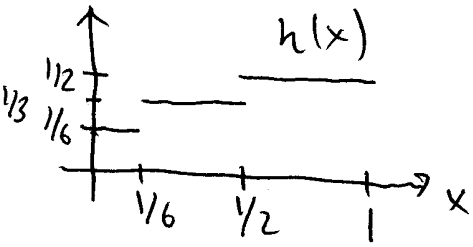
a)



b)



c)



d) There are four degrees of freedom $\Rightarrow \dim V_h = 4$

Exe 1.6.3

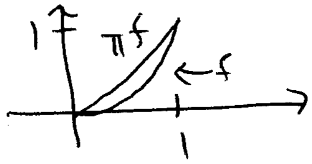
Let $I = [0,1]$. Determine the interpolant πf and plot πf and f for

a) $f = x^2$

b) $f = 3\sin(2\pi x)$

Sol

a) $\pi f = \lambda_0 f(0) + \lambda_1 f(1) = (1-x)\cdot 0 + x \cdot 1 = x$



b) $\pi f = \lambda_0 f(0) + \lambda_1 f(1) = (1-x)\cdot 0 + x \cdot 0 = 0$

Exe 1.6.4

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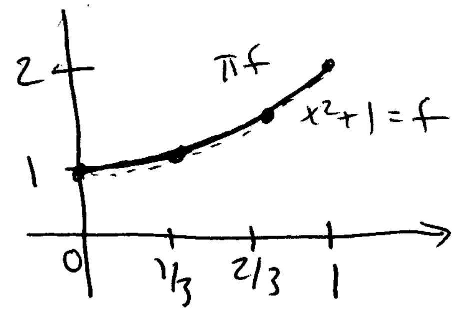
Let $I=[0,1]$ $\quad x_0=0$, $x_1=\frac{1}{3}$, $x_2=\frac{2}{3}$, $x_3=1$. Draw πf

a) $f=x^2+1$

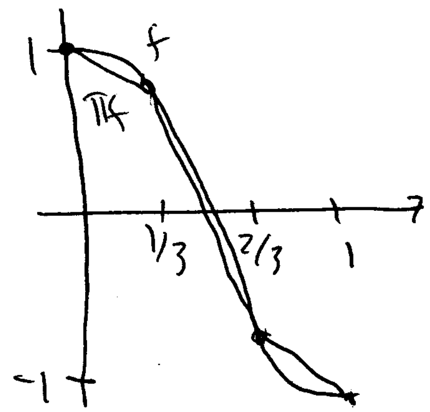
b) $f=\cos\pi x$

Sol

a)



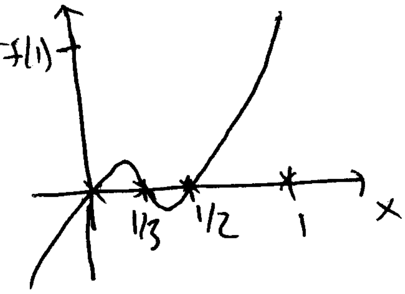
b)



Exe 1.6.5

Let $I=(0,1)$ $\quad$ Compute $\|f\|_{I,\infty}$, $f=x(x-\frac{1}{2})(x-\frac{1}{3})$

Sol



$f(1)=1\cdot\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$

we note that if $x\in[0,1]$

$f(x)=x(x-\frac{1}{2})(x-\frac{1}{3})\leq 1\cdot\frac{1}{2}\cdot\frac{2}{3}=\frac{1}{3}$

$\Rightarrow \|f\|_{I,\infty}=\frac{1}{3}$.

Exe 1.6.6 2009-01-26 AM

Let $I = (0,1)$, $f(x) = x^2$.

a) Calculate $\int_I f(x)dx$

b) Compute approx. using trapezoidal rule

c) —— " —— midpoint rule

d) Compare errors in b) and c).

Sol

a) $\int_0^1 x^2\,dx = 1/3$

b) $\int_0^1 x^2\,dx \approx (1-0)\frac{1^2+0^2}{2} = \frac{1}{2}$

c) $\int_0^1 x^2\,dx \approx (1-0)\left(\frac{0+1}{2}\right)^2 = \frac{1}{4}$

d) $|\text{error}_{tr}| = 1/6$

$|\text{error}_{mid}| = 1/12$

Both are exact for linear polynomials, $p=1$

Eq 1.15 gives $\left|\int_I f\,dx - Q_I(f)\right| \le C\,|1|^2 \left\|\frac{d^2f}{dx^2}\right\| = 2\cdot C$.

True if $C > \frac{1}{12}$.

Exe 1.6.8 2009-01-26 AM

Let $I = (0,1)$, $f(x) = x^2$, $x \in I$

a) Let $V_h = P_1(I)$. Calculate $P_h f$

b) Divide into two subintervals and calc. $P_h f$

c) Plot results and compare with πf.

Sol

a) Let $\varphi_0 = 1-x$, $\varphi_1 = x$, $P_h f = \alpha \varphi_0 + \beta \varphi_1$

α, β are determined by Eq 1.40 - 1.46

$m_{ij} = \int_0^1 \varphi_i \varphi_j$, $i,j = 0,1$ $\quad b_i = \int_0^1 f \cdot \varphi_i$, $i = 0,1$

$$M = \begin{bmatrix} \int_0^1 \varphi_0\varphi_0 & \int_0^1 \varphi_0\varphi_1 \\ \int_0^1 \varphi_1\varphi_0 & \int_0^1 \varphi_1\varphi_1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \qquad b = \begin{bmatrix} 1/12 \\ 1/4 \end{bmatrix}$$

$$\Rightarrow \alpha = -\frac{1}{6} , \beta = \frac{5}{6} \quad \therefore P_h f = -\frac{1}{6}\varphi_0 + \frac{5}{6}\varphi_1$$

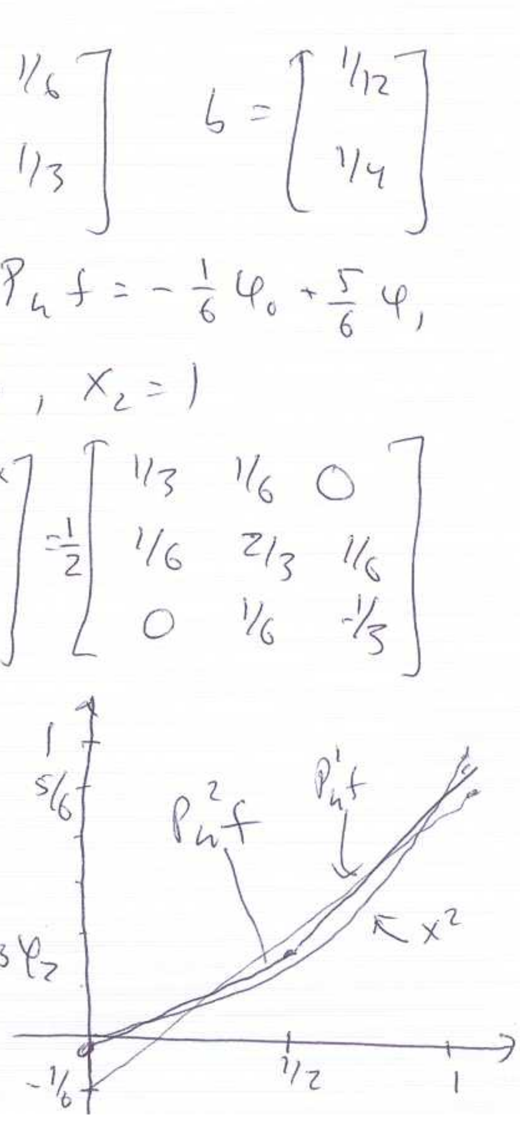
b) $I = (0,1)$ $\quad x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$

$$M = \begin{bmatrix} \int_0^1 \varphi_0\varphi_0 dx & \int_0^1 \varphi_1\varphi_0 dx & \int_0^1 \varphi_2\varphi_0 dx \\ \int_0^1 \varphi_0\varphi_1 dx & \int_0^1 \varphi_1\varphi_1 dx & \int_0^1 \varphi_2\varphi_1 dx \\ \int_0^1 \varphi_0\varphi_2 dx & \int_0^1 \varphi_1\varphi_2 dx & \int_0^1 \varphi_2\varphi_2 dx \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{bmatrix}$$

$$b = \begin{bmatrix} \int_0^{1/2} \varphi_0 x^2 dx \\ \int_0^1 \varphi_1 x^2 dx \\ \int_{1/2}^1 \varphi_2 x^2 dx \end{bmatrix} = \begin{bmatrix} 1/96 \\ 14/96 \\ 17/96 \end{bmatrix}$$

$$\Rightarrow P_h f = -0{,}1042\,\varphi_0 + 0{,}2083\,\varphi_1 + 0{,}9583\,\varphi_2$$

c)



Exe 1.6.9

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Show that $\int_\Omega (f - P_h f) v \, dx = 0 \quad \forall v \in V_h \iff$

$\int_\Omega (f - P_h f) \varphi_i \, dx = 0 \quad i = 0, \dots, N$ where $\{\varphi_i\}_{i=0}^N$ spans V_h.

Sol

$\Rightarrow$) Since $\varphi_i \in V_h$ we have

$$\int_\Omega (f - P_h f) \varphi_i = 0 \quad i = 0, \dots, N$$

$\Leftarrow$ Any $v \in V_h$ can be written $v = \sum_{i=0}^N \gamma_i \varphi_i$

$$\int_\Omega (f - P_h f) v \, dx = \sum_{i=0}^N \gamma_i \int_\Omega (f - P_h f) \varphi_i \, dx = 0.$$

Exe 1.6.10

Let $(f, g) = \int_I f \cdot g \, dx$, $\|f\|^2_{L^2(I)} = (f, f)$

Let $I = (0, \pi)$, $f = x$, $g = \cos x$, $h = 2\cos 3x$

a) Calc. (f, g)

b) Calc. (g, h) are g, h orthogonal?

c) Calc. $\|f\|$ and $\|g\|$.

Sol

a) $(f, g) = \int_0^\pi x \cdot \cos x \, dx \overset{IP}{=} [x \cdot \sin x]_0^\pi - \int_0^\pi \sin x \, dx = -2$

b) $(g, h) = \int_0^\pi 2\cos x \cdot \cos 3x \, dx = \text{IP twice} = 0$, yes!

c) $\|f\|_{L^2} = \left(\int_0^\pi x^2 \, dx \right)^{1/2} = \frac{\pi^{3/2}}{3^{1/2}}$

$\|g\|_{L^2} = \left(\int_0^\pi \cos^2 x \, dx \right)^{1/2} = \text{trig rule} = \left(\frac{\pi}{2} \right)^{1/2}$

Exe 1.6.11

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Let $V = \text{span}(\{v_i\}_{i=1}^m) \subset \mathbb{R}^n$, $P : \mathbb{R}^n \to V$ orthogonal proj. Derive eq for Px.
What happends if $\{v_1, \dots v_m\}$ is orth.?

Sol

$$Px = \sum_{i=1}^{m} \alpha_i \cancel{v}_i \quad , \alpha_i \in \mathbb{R}$$

P being orth. proj. $\Rightarrow (x - Px)^T v_j = 0 \quad j = 1, \dots, m$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i v_i^T v_j = x^T v_j \quad , j = 1, \dots, m$$

or
$$\begin{bmatrix} v_1^T v_1 & v_2^T v_{\cancel{1}} & \cdots & v_m^T v_1 \\ v_2^T v_1 & & & \\ \vdots & & & \\ \vdots & & & v_m^T v_m \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} x^T v_1 \\ x^T v_2 \\ \vdots \\ x^T v_m \end{bmatrix}$$

If v_i are orth $\Rightarrow$ the matrix is diagonal.

Exe 1.6.12

Show $\{1, x, \frac{3x^2-1}{2}\}$ is a basis for $P_2(-1,1)$
Compute and draw the L^2 proj $P_h f \in P_2(I)$
for a) $f(x) = 1 + 2x$
b) $f(x) = x^3$

Sol

$1, x, \frac{3x^2-1}{2}$ are linearly independent and have the same dim as $P_2(-1,1)$.

Since $1 + 2x \in P_2(-1,1)$, $P_h f = 1 + 2x$

Now let $f(x) = x^3$ 2009-01-26 AM

we note that $\int_{-1}^{1} 1 \cdot x \, dx = 0$, $\int_{-1}^{1} 1 \cdot \frac{3x^2-1}{2} dx = 0$,

$\int_{-1}^{1} x \cdot \frac{3x^2-1}{2} dx = 0 \Rightarrow$ orth. basis.

$\int_{-1}^{1} 1 \cdot 1 \, dx = 2$, $\int_{-1}^{1} x^2 dx = \frac{2}{3}$, $\int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 = \frac{2}{5}$

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/5 \end{bmatrix} \qquad b = \begin{bmatrix} \int_{-1}^{1} f \varphi_0 \\ \int_{-1}^{1} f \cdot \varphi_1 \\ \int_{-1}^{1} f \cdot \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \\ 0 \end{bmatrix}$$

$$\Rightarrow P_n f = \frac{3}{5} x.$$

Exe 1.6.13

Show $\{\varphi_j\}_{j=0}^{N}$ is almost orth?
How does this effect the mass matrix?
What if it was orthogonal?

Sol

$\int_I \varphi_i \cdot \varphi_j = 0$ if $|i-j| > 1 \Rightarrow$ "almost" orth.

The mass matrix i tridiagonal.

It would be diagonal if orth.

Exe 1.6.14

See code pg 19.