Chapter 2

2009-01-26 AM

Exe 2.7.1

let $\begin{cases} -u''_{xx} = f \\ u(0)=u(1)=0 \end{cases}$

solve analytically for

a) $f(x)=1$

b) $f(x)=x-u$

Sol

a) Assume $u(x) = ax^2+bx+c$

$-u'' = -2a = 1 \Rightarrow a = -\frac{1}{2}$

$u(0) = c = 0 \quad u(1) = -\frac{1}{2}+b = 0 \Rightarrow b = \frac{1}{2}$

$u(x) = \frac{1}{2}(1-x)\cdot x$.

b) $-u''_{xx} + u = x$.

Homogeneous solution $v(x) = ae^{x} + be^{-x}$

"Particular" solution $w(x) = x$

Let $u(x) = ae^{x} + be^{-x} + x$

$u(0) = ae^{0} + be^{-0} + 0 = a+b = 0 \Rightarrow a=-b$

$u(1) = ae + be^{-1} + 1 = b(e^{-1}-e^{1})+1 = 0 \Rightarrow b = \frac{-1}{e^{-1}-e}$

$u(x) = x - \frac{e^{x}-e^{-x}}{e^{1}-e^{-1}}$

Exe 2.7.2

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Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{2}$

Let $V_{h,0} = \{ v \in V_h : v(0) = v(1) = 0 \}$

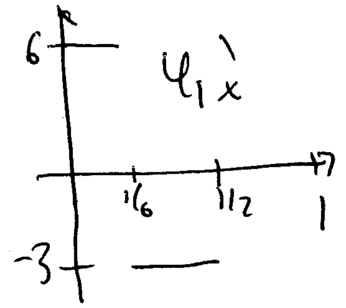
a) Compute stiffness matrix

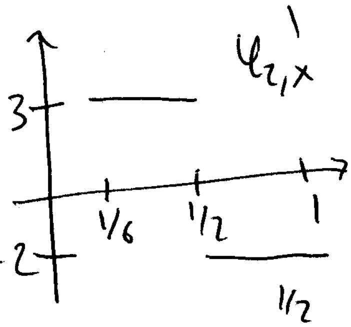
b) Compute Load vector for $f(x) = 1$

c) Solve system $A\xi = b$, plot U.

Sol

a) $a_{ij} = \int_0^1 \varphi_{j,x}' \varphi_{i,x}' \, dx \quad \forall\, i = 1,2$





$$a_{12} = a_{21} = \int_0^1 \varphi_{1,x}' \varphi_{2,x}' = \int_{1/6}^{1/2} (-9)\, dx = -3$$

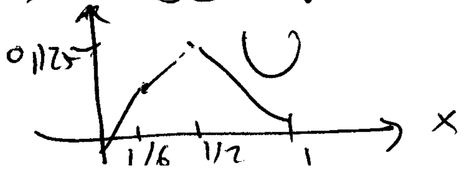
$$a_{11} = \int_0^{1/6} 6^2 \, dx + \int_{1/6}^{1/2} (-3)^2 = 9$$

$$a_{22} = \int_0^1 \varphi_{2,x}' \varphi_{2,x}' = \int_{1/6}^{1/2} 9 + \int_{1/2}^{1} 4 = 5$$

b) $f(x) = 1$, $b_1 = \int_0^1 1 \cdot \varphi_1 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$$b_2 = \int_0^1 1 \cdot \varphi_2 = \frac{5}{12}$$

c) $$\begin{bmatrix} 9 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 5/12 \end{bmatrix} \Rightarrow \xi = \begin{bmatrix} 0{,}0694 \\ 0{,}125 \end{bmatrix}$$



Exe 2.7.3

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Consider $-u''_{xx} = 7, \quad x \in (0,1)$

$u(0)=2, u(1)=3$

a) Determine V_h

b) Formulate the FEM

c) Derive discrete system with 4 nodes (uniform)

Sol

a) Let $x_j = \frac{j}{N}, \ j=0,\dots,N$, let $\varphi_i(x_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ be the hat functions. Then $V_h = \text{span}(\{\varphi_i\}_{i=0}^{N})$

b) We use Robin bnd cond with $R_0 = R_1 = 10^6$ to approximate Dirichlet (Eq 2.41) (This is not the only possible choice, $V_{h,0}$ can also be used).

We get, following 2.42-2.44, 2.46, Find $U \in V_h$ s.t

$$\int_0^1 U_x v_x dx + 10^6 U(1)v(1) + 10^6 U(0)v(0) = \int_0^1 7 \cdot v dx + 3 \cdot 10^6 v(1) + 2 \cdot 10^6 v(0)$$
$$\forall v \in V_h$$

c) Let $x_0 = 0, \ x_1 = \frac{1}{3}, \ x_2 = \frac{2}{3}, \ x_3 = 1$

$$a_{ij} = \int_0^1 \varphi_{i,x}' \varphi_{j,x}' dx + 10^6 \varphi_i(0)\varphi_j(0) + 10^6 \varphi_i(1)\varphi_j(1)$$

$$b_j = \int_0^1 7 \cdot \varphi_j dx + 3 \cdot 10^6 \varphi_j(1) + 2 \cdot 10^6 \varphi_j(0)$$

$$\varphi_{i,x}' = \begin{cases} 3 & x \in [x_{i-1}, x_i] \\ -3 & x \in [x_i, x_{i+1}] \end{cases}$$

$$A = \begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 2 \cdot 10^6 & & & \\ & & & \\ & & & \\ & & & 3 \cdot 10^6 \end{bmatrix}$$

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$$b_0 = \int_0^1 7 \cdot \varphi_0 \, dx + 3 \cdot 10^6 \cancel{\varphi_0(1)} + 2 \cdot 10^6 \underbrace{\varphi_0(0)}_{1} =$$

$$= \frac{7}{6} + 2 \cdot 10^6$$

$$b_1 = \int_0^1 7 \cdot \varphi_1 + 0 + 0 = \frac{7}{3}$$

$$b_2 = b_1$$

$$b_3 = \int_0^1 7 \cdot \varphi_3 + 10^6 3 \cdot \overbrace{\varphi_3(1)}^{1} + 2 \cdot 10^6 \cancel{\varphi_3(0)} = \frac{7}{6} + 3 \cdot 10^6$$

$$b = \begin{bmatrix} 7/6 + 2 \cdot 10^6 \\ 7/3 \\ 7/3 \\ 7/6 + 3 \cdot 10^6 \end{bmatrix} \qquad A\xi = b$$

$$U = \sum_{i=0}^{3} \xi_i \varphi_i$$

Exe 2.7.4

Consider $\quad -((1+x)u_x')_x' = 0 \quad x \in (0,1)$

$$u(0) = u'(1) = 1$$

let $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$

$V_h = \text{span}(\{\varphi_i\}_{i=1}^{3})$ not 0 since $V_h(0) = 0$

a) Derive analytica sol.

b) Formulate FEM

c) Verify $A = \frac{1}{2}\begin{bmatrix} 16 & -9 & 0 \\ -9 & 20 & -11 \\ 0 & -11 & 11 \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

d) Verify A pos. def.

Sol

a) Integrate $-(1+x)u_x' = C$ const

$u_x' = \frac{-c}{1+x} \Rightarrow u(x) = -c \ln(1+x) + D$

$u(0) = 0 \Rightarrow D = 0 \quad u'(x) = \frac{-c}{1+x}, u'(1) = -\frac{c}{2} = 1$

$\Rightarrow c = -2 \quad u(x) = 2\ln(1+x)$

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b) FEM Find $U \in V_h$ s.t.

$$\int_0^1 (1+x)\, U'_x \cdot v'_x \, dx = \overset{1+x=2}{2} \cdot \overset{u'(1)=1}{1} \cdot v(1) \quad \forall v \in V_h .$$

c) $b_j = 2\varphi_j(1) \Rightarrow b = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$$A_{i,i} = 9\int_{x_{i-1}}^{x_i} (1+x)\,dx + 9\int_{x_i}^{x_{i+1}} (1+x)\,dx = \begin{cases} 8 & i=1 \\ 10 & i=2 \\ 5.5 & i=3 \end{cases}$$

$$A_{i,i+1} = \frac{-9}{1}\int_{x_i}^{x_{i+1}} 1+x \, dx = \begin{cases} -9 & i=1 \\ -11 & i=2 \end{cases}$$

$A_{i+1,i} = A_{i,i+1}$ $\quad A_{i,j} = 0 \quad |i-j|>1$

d) Positive eigenvalues.

<u>Exe 2.7.5</u>

$$(*) \quad \begin{aligned} -u''_{xx} &= f \qquad x \in (0,1) \\ u'_x(0) &= u'_x(1) = 0 \end{aligned}$$

$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1$

Compute stiffness matrix, why is it singular?

<u>Sol</u>

$$A = 2\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$\det(A) = 0 \Rightarrow$ singular. Note that if u solves $(*)$ so does $u + c$, where c is const, hence the solution in not unique.

Exe 2.7.6

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$$-u''_{xx} + u = f \quad x \in (0,1)$$

$$u(0) = u(1) = 0$$

a) Choose V_h

b) Formulate FEM

c) Derive disc system of eqs.

Sol

a) Let $x_j = \frac{j}{N}$, $j = 0, \dots, N$. Let $V_h = V_{h,0} = \operatorname{span}(\{\varphi_i\}_{i=1}^{N-1})$.

b) Find $U \in V_{h,0}$ s.t

$$\int_0^1 U'_x v'_{h,x} dx + \int_0^1 U \cdot v_h dx = \int_0^1 f \cdot v_h dx \quad \forall v_h \in V_{h,0}$$

c) $b_j = \int_0^1 f \cdot \varphi_j dx$, $j = 1, \dots, N-1$

$$a_{i,j} = \int_0^1 \varphi_{i,x}' \varphi_{j,x}' dx + \int_0^1 \varphi_i \varphi_j dx$$

L^2 proj chapter

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} + h \begin{bmatrix} 2/3 & 1/6 & & \\ 1/6 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & 2/3 \end{bmatrix}$$

$i, j = 1, \dots, N-1$

Exe 2.7.7

Let $I=(0,1)$ $u(0)=0$

Show $\|u\|_{L^2(I)} \le C\|u_x'\|_{L^2(I)}$ Poincare inequality

Proof

$$\|u\|^2_{L^2(0,1)} = \int_0^1 u(x)^2 dx = \int_0^1 \left(u(0) + \int_0^x u'(y)dy \right)^2 dx$$

$$= \int_0^1 \left(\int_0^x 1\cdot u'(y)dy \right)^2 dx \le \int_0^1 \left(\left(\int_0^1 1\cdot u'(y)dy \right) \right)^2 dx$$

$$\le \int_0^1 \left(\int_0^1 1^2 dx \right)\left(\int_0^1 u'(y)^2 dy \right) dx = \|u'\|^2_{L^2(0,1)}$$

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Exe 2.7.8

$$\begin{cases} -u''_{xx} + u = f, & x \in I = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Derive A posteriori error estimate.

Sol G.O $\int_0^1 e_x' v_x' dx + \int_0^1 e\cdot v = 0 \quad \forall v \in V_{h,0}$

where $e = u - U$.

$$\int_0^1 e_x'\cdot e_x' dx + \int_0^1 e\cdot e dx = \int_0^1 f\cdot e - \int_0^1 U_x' e_x' - \int_0^1 U\cdot e$$

$$= \int_0^1 f(e-\pi e) - \int_0^1 U_x'(e-\pi e)_x' - \int_0^1 U\cdot(e-\pi e) =$$

$$= \int_0^1 (f + U''_{xx} - U)(e-\pi e) = \sum_{i=1}^{M} \int_{x_{i-1}}^{x_{i+1}} (f + U''_{xx} - U)(e-\pi e)$$

$$\le C \sum_{i=1}^{M} \int_{x_{i-1}}^{x_{i+1}} |f + U''_{xx} - U|\cdot|e-\pi e| dx$$

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$$\leq C \sum_{i=1}^{N} h_i \| U''_{xx} + f - U \|_{L^2(I_i)} \cdot \| e'_x \|_{L^2(I_i)}$$

$$\leq C \left(\sum_{i=1}^{N} h_i^2 \| U''_{xx} + f - U \|^2_{L^2(I_i)} \right)^{1/2} \| e'_x \|_{L^2(I)}$$

$$\leq C \left(\sum_{i=1}^{N} h_i^2 \| U''_{xx} + f - U \|^2_{L^2(I_i)} \right)^{1/2} \left(\| e'_x \|^2_{L^2(I)} + \| e \|^2_{L^2(I)} \right)^{1/2}$$

Divide both sides by $\left(\| e'_x \|^2_{L^2} + \| e \|^2_{L^2} \right)^{1/2}$

$$\Rightarrow \left(\| e_x \|^2_{L^2(I)} + \| e \|^2_{L^2(I)} \right)^{1/2} \leq C \left(\sum_{i=1}^{N} h_i^2 \| U''_{xx} + f - U \|^2_{L^2(I_i)} \right)^{1/2}$$

Exe 2.7.9

Consider $-\varepsilon u''_{xx} + x u' + u = f \quad , x \in I = (0,1)$

$$u(0) = u'_x(1) = 0$$

$\varepsilon > 0$ const , prove $\| \varepsilon u_{xx} \|_{L^2(I)} \leq \| f \|_{L^2(I)}$

Sol

Multiply eq with $-\varepsilon u''_{xx}$ and integrate

$$\| \varepsilon u''_{xx} \|^2 \cancel{+} \int_0^1 x u' \varepsilon u''_{xx} \, dx \cancel{-} \int_0^1 u \cdot \varepsilon u''_{xx} \, dx = -\int_0^1 f \cdot \varepsilon u''_{xx} \, dx$$

$$\leq \| f \|_{L^2(I)} \cdot \| \varepsilon u''_{xx} \|_{L^2(I)}$$

$$\left(\int_0^1 x u'_x (-\varepsilon u''_{xx}) \, dx + \int_0^1 u \cdot -\varepsilon u''_{xx} \, dx = \int_0^1 (x u'_x)'_x \, \varepsilon u'_x - \cancel{\left[x u'_x \cdot \varepsilon u'_x \right]_0^1} \right.$$

$$\left. + \int_0^1 \varepsilon u'_x \cdot u'_x \, dx = 2 \int_0^1 \varepsilon u'_x \cdot u'_x \, dx + \int_0^1 \varepsilon x u''_{xx} \cdot u'_x \, dx \right)$$

$$\Rightarrow \int_0^1 x u'_x (-\varepsilon u''_{xx}) \, dx = \frac{1}{2} \int_0^1 \varepsilon u'_x \cdot u'_x \, dx$$

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$$\therefore \quad \|\varepsilon u''_{xx}\|^2 + \frac{3}{2}\|\sqrt{\varepsilon}u'_x\|^2 \le \|f\|\cdot\|\varepsilon u''_{xx}\|$$

$$\Rightarrow \quad \|\varepsilon u''_{xx}\| \le \|f\|.$$

Exe 2.7.10

Let $F(w) = \frac{1}{2}\int_I w_x^2 - \int_I f\cdot w$ and let

$$\int_I u'_x v'_x = \int_I f\cdot v \quad \forall v \in V_0$$

Show $F(u) \le F(w) \quad \forall w \in V_0$.

Sol

$$F(w) = F(u+v) = \frac{1}{2}\int_I (u+v)_x^2 - \int_I f\cdot(u+v) =$$

$$= F(u) + \int_I u'_x v'_x - \int_I f\cdot v + \frac{1}{2}\int_I v_x^2\,dx$$

$$= F(u) + \underbrace{\frac{1}{2}\int_I v_x^2\,dx}_{\geq 0}. \Rightarrow F(u) \le F(w) \quad \forall w \in V_0.$$