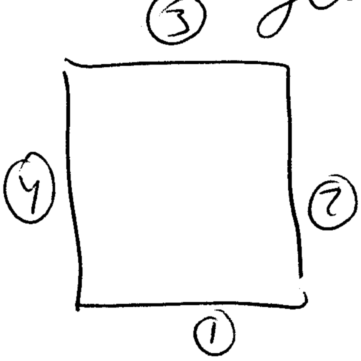
Chapter 3

Exe 3.7.1

Write down geom for unit square

Sol



$$geom = \begin{array}{c l} \left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} \leftarrow \text{line segments} \\ \leftarrow x_1 \\ \leftarrow x_2 \\ \leftarrow y_1 \\ \leftarrow y_2 \\ \leftarrow \text{left domain} \\ \leftarrow \text{right domain} \end{array} \end{array}$$

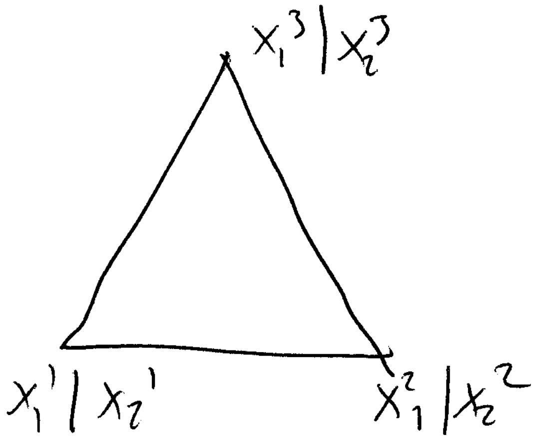
Exe 3.7.2

Express area of arbitrary triangle
(x_1^1, x_2^1) (x_1^2, x_2^2) (x_1^3, x_2^3)

Sol

$$area = \frac{1}{2} |v_{31} \times v_{21}|$$ where

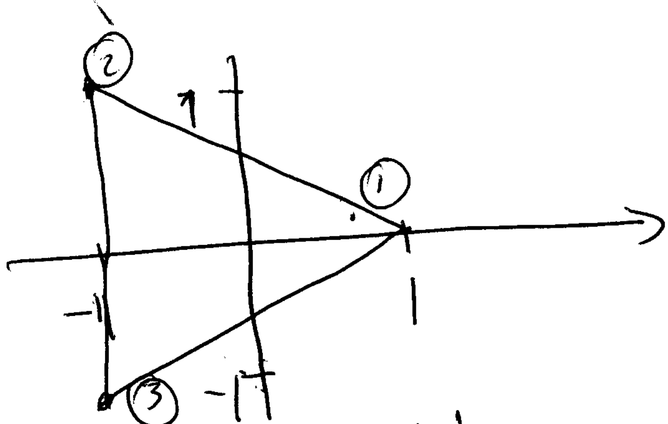
$$v_{31} = (x_1^3 - x_1^1, x_2^3 - x_2^1)$$
$$v_{21} = (x_1^2 - x_1^1, x_2^2 - x_2^1)$$



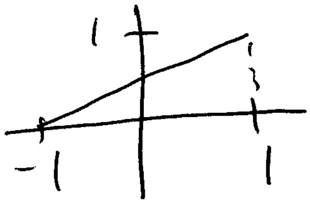
Exe 3,7.3

Derive basis fcn for piecewise linears on the triangle $(-1,-1)$, $(1,0)$, $(-1,1)$

Sol



$\varphi_1)$ let $y = 0$



$$\varphi_1 = \frac{1}{2}(1+x)$$

$\varphi_2)$ $\varphi_2 = ax + by + c$, $\varphi_2(-1,1) = -a+b+c = 1$

$\varphi_2(1,0) = a + c = 0 \Rightarrow a = -c$

$\varphi_2(-1,-1) = -a-b+c = 0 \Rightarrow b = 2c$

$$\Rightarrow c = \frac{1}{4} \quad a = -\frac{1}{4} \quad b = \frac{1}{2}$$

$$\varphi_2 = -\frac{1}{4}x + \frac{1}{2}y + \frac{1}{4}$$

$\varphi_3)$

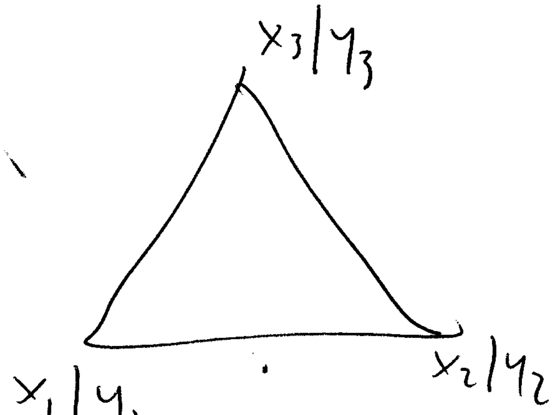
$$\varphi_3 = 1 - \varphi_1 - \varphi_2 = 1 - \frac{1}{2} - \frac{x}{2} + \frac{1}{4}x - \frac{1}{2}y - \frac{1}{4}$$

$$= \frac{1}{4} - \frac{x}{4} - \frac{1}{2}y$$

Exe 3.7.4

Determine basis fcns for arbitrary triangle.

Sol



$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$ Let $\varphi_1 = ax + by + c$

$$\Rightarrow \quad \varphi_1(x_1, y_1) = ax_1 + by_1 + c = 1$$
$$\varphi_1(x_2, y_2) = ax_2 + by_2 + c = 0$$
$$\varphi_1(x_3, y_3) = ax_3 + by_3 + c = 0$$

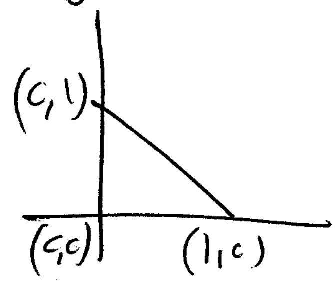
Solve system!

Defines $\varphi_1 = ax + by + c$

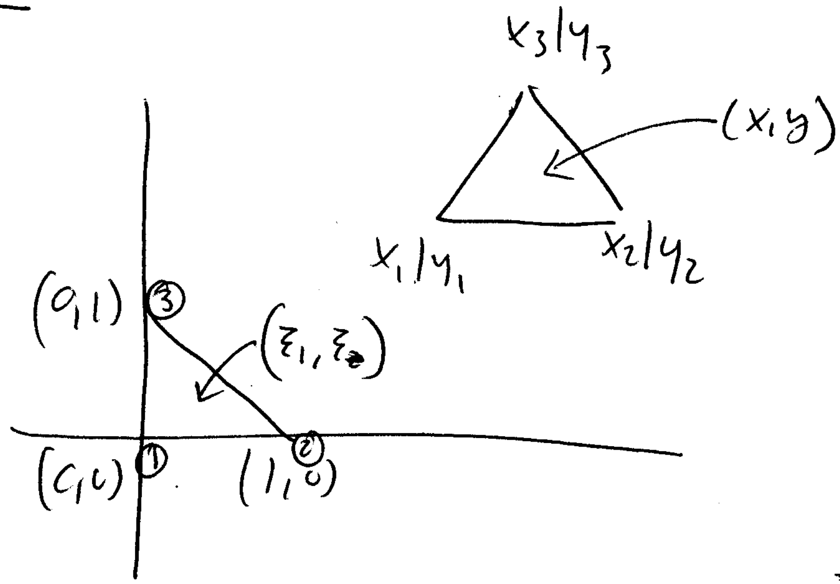
φ_2 & φ_3 are defined in the same way.

Exe 3.7.5

Determine map from arbitrary triangle to reference triangle $\bar{K}$.



Sol



basis on $\bar{K}$

Eq. 3.63 gives $(x,y) = T(\xi_1,\xi_2) = \sum_{i=1}^{3} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \varphi_i(\xi_1,\xi_2)$

$= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}(1-\xi_1-\xi_2) + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\xi_1 + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\xi_2$

$= \begin{bmatrix} x_2-x_1 & x_3-x_1 \\ y_2-y_1 & y_3-y_1 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

invertable since $|K|>0$ det nonz

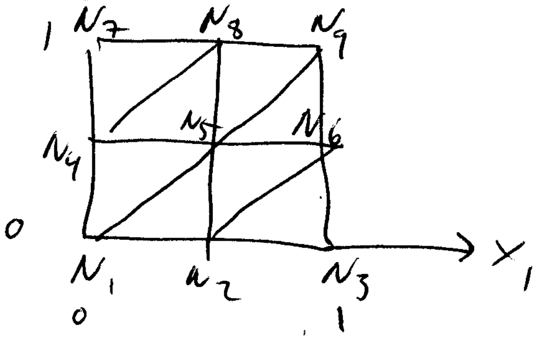
$\Rightarrow (x,y) = A(\xi_1,\xi_2) + (x_1, x_2)$

$\Rightarrow A(\xi_1,\xi_2) = (x-x_1, y-y_1)$

$(\xi_1,\xi_2) = A^{-1}(x-x_1, y-y_1)$

Exe 3.7.6

Given triangulation



a) Write down p, t

b) Determine mesh fcn $h(x)$

Sol

a)
$$p = \begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \end{bmatrix}$$

$$t = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 5 & 5 \\ 2 & 5 & 6 & 3 & 5 & 8 & 6 & 9 \\ 5 & 4 & 5 & 6 & 8 & 7 & 9 & 8 \end{bmatrix}$$

b) (Same area for all triangles) same longest edge

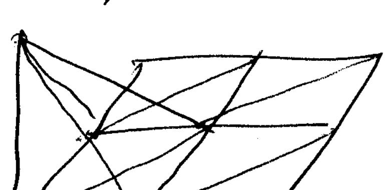
$$h(x) = \frac{1}{\sqrt{2}}$$

Exe 3.7.6

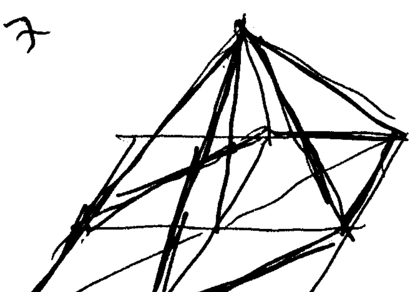
Draw φ_1, φ_5 in Fig 3.7

Sol

φ_1



φ_5



Exe 3.7.8

- Consider same mesh as before

a) Determine sparsity

b) Compute $\int_\Omega \phi_1 \phi_2 \, dx$, $\int_\Omega \phi_7 \cdot \phi_4 \, dx$, $\int_\Omega \phi_7 \cdot \phi_8$, $\int_\Omega x_1 \phi_1$

Sol

a)
$$\begin{bmatrix} \times & \times & & \times & \times & & & & \\ \times & \times & \times & & \times & \times & & & \\ & \times & \times & & & \times & & & \\ \times & & & \times & \times & & \times & \times & \\ \times & \times & & \times & \times & \times & & \times & \times \\ & \times & \times & & \times & \times & & & \times \\ & & & \times & & & \times & \times & \\ & & & \times & \times & & \times & \times & \times \\ & & & & \times & \times & & \times & \times \end{bmatrix}$$

b) We use eq 3.76 $\int_K \phi_i \phi_j \, dx = \frac{1}{12} \cdot |K| = \frac{1}{96}$

$$\Rightarrow \int_\Omega \phi_1 \phi_2 = \text{overlap on one } K = \frac{1}{96}$$

$$\int_\Omega \phi_7 \cdot \phi_4 = \text{— " —} = \frac{1}{96}$$

$$\int_\Omega \phi_7 \cdot \phi_8 = \text{— " —} \quad \frac{1}{96}$$

$$\int_\Omega x_1 \cdot \phi_1 \, dx = \int_{K_\triangledown} x_1 \cdot \phi_1 + \int_{K_\vartriangle} x_1 \cdot \phi_1 = \int_{K_\triangledown} \left(\tfrac{1}{2}\phi_4 + \tfrac{1}{2}\phi_5\right) \phi_1$$

$$+ \int_{K_\vartriangle} \tfrac{1}{2} \phi_5 \cdot \phi_1 = \left\{ \begin{array}{l} \int_\square \phi_5 \phi_1 = 2 \cdot \frac{1}{12} \cdot \frac{1}{8} = \frac{1}{48} \\ \int_\triangledown \phi_4 \cdot \phi_1 = \frac{1}{96} \end{array} \right\}$$

$$= \frac{1}{2} \cdot \frac{1}{96} + \frac{1}{2} \cdot \frac{1}{48} = \frac{3}{2} \cdot \frac{1}{96} = \frac{1}{64}$$

Exe 3.7.9

Let $f = x_1 \cdot x_2$, $\Omega = [0,1]^2$

a) $\int_\Omega f \, dx$

b) Use centre of gravity with mesh from 3.7

c) Use corner quad. — " — 3.7

Sol

a) $\int_0^1 \int_0^1 x_1 \cdot x_2 \, dx_1 dx_2 = \frac{1}{4}$

b) $I = \sum_{i=1}^{8} f(\bar{x}_i) |K| = \sum_{i=1}^{8} f(\bar{x}_i) \frac{1}{8} = 0{,}2431$ Matlab

c) $I = 0{,}2708$ Matlab.

Exe 3.7.10

$f = x_1^2$ Compute $P_h f \in V_h$ mesh from 3.7

Sol Let $M_{ij} = \int_\Omega \varphi_i \cdot \varphi_j \, dx$ $b_j = \int_\Omega x_1^2 \cdot \varphi_j \, dx$

$P_h f = \sum_{i=1}^{9} \alpha_i \varphi_i$ where $M \vec{\alpha} = b$

Use matlab code pg 61-62.