Chapter 4

Exe 4.9.1

Prove the Cauchy-Schwartz inequality $\left|\int_\Omega u \cdot v\,dx\right| \leq \|u\| \cdot \|v\|$ where $\|v\|^2 = \int_\Omega v^2 dx$

Proof: Classical trick, assume $\|v\| \neq 0$ (otherwise trivial)

$$0 \leq \|u - \lambda v\|^2 = \|u\|^2 - 2\lambda \int_\Omega u \cdot v\,dx + \lambda^2 \|v\|^2$$

Now let $\lambda = \dfrac{\int_\Omega u \cdot v\,dx}{\|v\|^2} \Rightarrow$

$$0 \leq \|u\|^2 - 2\frac{(\int_\Omega u \cdot v\,dx)^2}{\|v\|^2} + \frac{(\int_\Omega u \cdot v\,dx)^2}{\|v\|^2}$$

multiply with $\|v\|^2$

$$0 \leq \|u\|^2 \cdot \|v\|^2 - \left(\int_\Omega u \cdot v\,dx\right)^2$$

$$\Rightarrow \left|\int_\Omega u \cdot v\,dx\right| \leq \|u\| \cdot \|v\|$$

Exe 4.9.2

Verify that $\|\nabla v\|$ is a norm on V_0.

Sol

(i) $\|v\| = 0 \iff v = 0$

(ii) $\|a \cdot v\| = |a| \cdot \|v\|$, $a \in \mathbb{R}$

(iii) $\|v + w\| \leq \|v\| + \|w\|$

(ii) and (iii) follows immediately since ∇ is linear $a\nabla u = \nabla a u$, $a \in \mathbb{R}$

$\nabla(u+v) = \nabla u + \nabla v$

and that $\|\cdot\|$ is a norm.

(i) If $u=0 \Rightarrow \nabla u = 0 \Rightarrow \|\nabla u\| = 0$

If $\|\nabla u\| = 0 \Rightarrow \nabla u = 0 \Rightarrow u = \text{constant}$ but $u \in V_0 \Rightarrow 0$ on bndry $\Rightarrow u = 0$.

Exe 4.9.3

Determine f so that $u = x(1-x)y(1-y)$ is a solution to $-\Delta u = f$ on $\Omega = [0,1]^2$ with $u=0$ on $\partial\Omega$. Compute ∇u, $\|u\|$, and $\|\nabla u\|$.

Sol

$$-\Delta u = -\Delta\, x(1-x)y(1-y) = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(x(1-x)y(1-y)\right)$$

$$= -y(1-y)\frac{\partial^2}{\partial x^2}(x-x^2) - x(1-x)\frac{\partial^2}{\partial y^2}(y-y^2) = 2x(1-x) + 2y(1-y)$$
$$= f(x,y)$$

$$\nabla u = \begin{bmatrix} y(1-y)(1-2x) \\ x(1-x)(1-2y) \end{bmatrix}$$

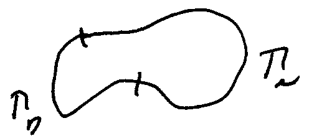
$$\|u\|^2 = \int_0^1\int_0^1 x^2(1-x)^2y^2(1-y)^2\,dx\,dy = \int x^2(1-x)^2dx \int y^2(1-y)^2 dy$$
$$= \left(\int_0^1 y^2 - 2y^3 + y^4\,dy\right)^2 = \left(\frac{1}{30}\right)^2 \Rightarrow \|u\| = \frac{1}{30}$$

$$\|\nabla u\|^2 = 2\int_0^1\int_0^1 [y(1-y)(1-2x)]^2\,dx\,dy = 2\int_0^1 y^2(1-y)^2 dy \int_0^1 (1-2x)^2 dx$$
$$= \frac{2}{30}\int_0^1 (1-2x)^2 dx = \frac{1}{15}\int_0^1 1 - 4x + 4x^2\,dx = \frac{1}{15}\left(1 - 2 + 4\cdot\frac{1}{3}\right)$$
$$= \frac{1}{45}$$

Exe 4.9.4

What are appropriate test and trailspace for

$$\begin{cases} -\Delta u = 0 & x \in \Omega \\ u = 0 & x \in \Gamma_D \\ n \cdot \nabla u = g & x \in \Gamma_N \end{cases}$$



Sol

One sol is $V_{0,D} = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega), \ v|_{\Gamma_D} = 0 \}$

both as test space $v \in V_{0,D}$, and trailspace $u \in V_{0,D}$

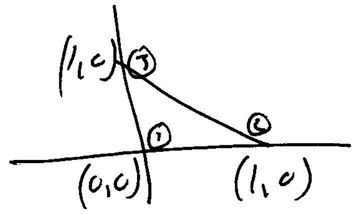
Another sol i $n \cdot \nabla u = \kappa(\tilde{g} - u) + g$ where $\kappa = 0$ on Γ_N

$\kappa = 10^6$ on Γ_D approximation with

$$V = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \}.$$

Exe 4.9.5

Compute element (local) stiffness and mass matrix on reference triangle



Sol

$\varphi_1 = 1 - x - y$, $\varphi_2 = x$, $\varphi_3 = y$

$\nabla\varphi_1 = (-1,-1)$, $\nabla\varphi_2 = (1,0)$, $\nabla\varphi_3 = (0,1)$, $|K| = \frac{1}{2}$

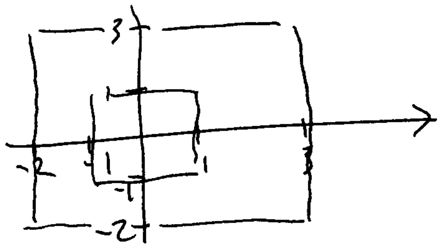
$$A^K_{ij} = \int_K \nabla\varphi_i \cdot \nabla\varphi_j = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$M^k_{i,j} = \{\text{eq. } 3.76\} = \int_F \varphi_i \cdot \varphi_j = \frac{1}{2}\begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{bmatrix}$$

Exe 4.9.6

let $\Omega = [-2,3]^2 \setminus [-1,1]^2$, construct geom.

Sol



$$geom = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 3 & 3 & -2 & -1 & 1 & 1 & -1 \\ 3 & 3 & -2 & -2 & 1 & 1 & -1 & -1 \\ -2 & -2 & 3 & 3 & -1 & -1 & 1 & 1 \\ -2 & 3 & 3 & -2 & -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Exe 4.9.8

Show that $\|\nabla u\| \le C\|f\|$ for $\begin{cases} -\Delta u = f \\ u = 0 \end{cases}$

Sol $\|\nabla u\|^2 = \int_\Omega \nabla u \cdot \nabla u \, dx = \int_\Omega -\Delta u \cdot u \, dx + \int_{\partial\Omega} n \cdot \nabla u \cdot u =$

$= \int_\Omega f \cdot u \, dx \le \|f\| \cdot \|u\| \overset{PF}{\le} C\|f\| \cdot \|\nabla u\|$

$\Rightarrow \|\nabla u\| \le C\|f\|$ ∎

Exe 4.9.9

let $\begin{cases} -\Delta u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D \\ n \cdot \nabla u = g \text{ on } \Gamma_N \end{cases}$ show $\|\nabla u\| \le C\|g\|_{L^2(\Gamma_N)}$

Sol $\|\nabla u\|^2 = \int_\Omega \nabla u \cdot \nabla u \, dx = \int_\Omega -\Delta u \cdot u \, dx + \int_\Gamma n \cdot \nabla u \cdot u = \int_{\Gamma_N} g \cdot u$

$\le \|g\|_{L^2(\Gamma_N)} \cdot \|u\|_{L^2(\Gamma_N)} \le \|g\|_{L^2(\Gamma_N)} C'(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)})$

$\overset{P.F}{\le} C'\|g\|_{L^2(\Gamma_N)} C''\|\nabla u\|_{L^2(\Omega)} \Rightarrow \|\nabla u\| \le C\|g\|_{L^2(\Gamma_N)}$ ∎

Exe 4.9.11

Prove $\|\nabla(u-U)\|^2 = \|\nabla u\|^2 - \|\nabla U\|^2$, for $\begin{cases} -\Delta u = f \\ u = 0 \end{cases}$

Proof: $\|\nabla(u-U)\|^2 = (\nabla u - U, \nabla u - U) = \|\nabla u\|^2 - \|\nabla U\|^2$

$- 2\int_\Omega \nabla u \cdot \nabla U\,dx + 2\int_\Omega \nabla U \cdot \nabla U\,dx =$

$= \|\nabla u\|^2 - \|\nabla U\|^2 - 2\int_\Omega \nabla(u-U)\nabla U\,dx =$

$= \|\nabla u\|^2 - \|\nabla U\|^2$

Exe 4.9.13

Consider $-\Delta u + u = f \quad , x \in \Omega$
$u = 0 \quad , x \in \partial\Omega$

a) Make variational form (weak form)

b) Formulate FEM

c) Show G.O

d) Show a priori estimate $\|\nabla u - U\|^2 + \|u-U\|^2 \leq C\sum_{K\in K} h_K^2 \|D^2 u\|^2_{L^2(K)}$

Sol a) let $V_0 = \{v : \|v\| + \|\nabla v\| < \infty\}$

Find $u \in V_0$ s.t $\int_\Omega \nabla u \cdot \nabla v\,dx + \int_\Omega u \cdot v\,dx = \int_\Omega f \cdot v\,dx, \forall v \in V$

b) let $V_{h,0} \subset V_0$ piecewise linear cont s.t $v|_{\partial\Omega} = 0$

Find $U \in V_{h,0}$ s.t $\int_\Omega \nabla U \cdot \nabla v\,dx + \int_\Omega U \cdot v\,dx = \int_\Omega f \cdot v\,dx \; \forall v \in V_h$

c) Subtract the two

$\int_\Omega \nabla(u-U) \cdot \nabla v\,dx + \int_\Omega (u-U)v\,dx = 0 \; \forall v \in V_{h,0}$

d) $\|\nabla(u-U)\|^2 + \|u-U\|^2 = \int_\Omega \nabla(u-U)\nabla u - U + (u-U)(u-U)\,dx$

$\overset{G.O}{=} \int_\Omega \nabla(u-U)\,\nabla(u-\pi u) + (u-U)(u-\pi u)\,dx =$

$\leq \|\nabla(u-U)\| \cdot \|\nabla(u-\pi u)\| + \|u-U\| \cdot \|u-\pi u\|$

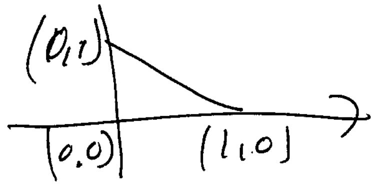
$\leq \frac{1}{2}\|\nabla u - \nabla U\|^2 + \frac{1}{2}\|\nabla u - \pi u\|^2 + \frac{1}{2}\|u-U\|^2 + \frac{1}{2}\|u-\pi u\|^2$

$\Rightarrow \|\nabla u - \nabla U\|^2 + \|u-U\|^2 \leq \|\nabla(u-\pi u)\|^2 + \|u-\pi u\|^2 \leq$

$\leq C\sum_{K\in K} h_K^2 \|D^2 u\|^2_{L^2(K)}$ interp.

Exe 4.9.12

Reference triangle $\bar{K}$



let $I(r,s) = \int_{\bar{K}} x_1^r x_2^s \, dx$

Show $I(r-1,s+1) = \frac{s+1}{r} I(r,s)$, $I(r,0) = \frac{1}{(r+1)(r+2)}$
and by induction $I(r,s) = \frac{r!\,s!}{(r+s+2)!}$

Sol

$$I(r,0) = \int_{\bar{K}} x_1^r \, dx_1 dx_2 = \int_0^1 \int_0^{1-x_1} x_1^r \, dx_2 dx_1 =$$

$$= \int_0^1 x_1^r (1-x_1) \, dx_1 = \frac{1}{r+1} - \frac{1}{r+2} = \frac{1}{(r+1)(r+2)}$$

$$I(r-1,s) = \int_0^1 x_1^{r-1} \left(\int_0^{1-x_1} x_2^s \, dx_2 \right) dx_1 = \int_0^1 x_1^{r-1} \frac{(1-x_1)^{s+1}}{s+1} \, dx_1$$

$$I(r,s-1) = \int_0^1 x_1^r \frac{(1-x_1)^s}{s} \, dx_2 = \int_0^1 r x_1^{r-1} \frac{(1-x_1)^{s+1}}{s(s+1)} \, dx_1$$

$$\Rightarrow I(r-1,s) = \frac{s}{r} I(r,s-1)$$

$$\Rightarrow I(r-1,1) = \frac{1}{r} I(r,0) = \frac{1}{r} \frac{1}{(r+1)(r+2)}$$

$$I(r-2,2) = \frac{2}{r-1} I(r-1,1) = \frac{2}{(r-1)\,r\,(r+1)(r+2)}$$

$$\vdots$$

$$I(r-p,p) = \frac{p!}{(r-(p-1))\cdot \ldots \cdot (r+2)}$$

let $r-p=t$

$$I(t,p) = \frac{p!}{(t-1)\cdot \ldots \cdot (t+p+2)} = \frac{p!\,t!}{(t+p+2)!}$$