Chapter 5

Exe 5.4.1

Consider $\begin{cases} \dot{u} - \Delta u + cu = f & \text{i } [0,1]\times[0,1] \\ u = 0 & \\ u(\cdot,0) = u_0 & \end{cases}$

Choose f, u_0 s.t $u = e^{-t}\sin\pi x \sin\pi y$ if $c=0$.

Sol

$$\dot{u} - \Delta u + 0\cdot u = -u + 2\pi^2 u = \underbrace{(2\pi^2-1)e^{-t}\sin\pi x\sin\pi y}_{f(x,y,t)}$$

$$u_0 = u(\cdot,0) = \sin\pi x \sin\pi y.$$

Exe 5.4.2

Show $\|u\| \le \|u_0\| \ \forall t>0$ if $f=0$

Sol

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx = \int_\Omega \dot{u}\cdot u\, dx = \int_\Omega -cu\cdot u + \Delta u\cdot u\, dx$$

$$= -\|\sqrt{c}u\|^2 - \|\nabla u\|^2 \le 0$$

$\Rightarrow \|u\|^2$ decreasing $\Rightarrow \|u(t)\|^2 \le \|u(0)\|^2$.

Exe 5.4.3

Discretize (1) in space.

Sol

Find $u \in V_0 = \{v : \|v\| + \|\nabla v\| < \infty, v|_{\partial\Omega} = 0\}$

Weak form: $(\dot{u}, v) + (\nabla u, \nabla v) + (cu, v) = (f, v)$ $\forall v \in V_0$

FEM: Find $U \in V_{h,0} = \{v \in V_h : v|_{\partial\Omega} = 0\}$ s.t

$$(\dot{U}, v) + (\nabla U, \nabla v) + (cU, v) = (f, v) \quad \forall v \in V_{h,0}$$

let $U = \sum_{i=1}^{n} \xi_i(t) \varphi_i$, $v = \varphi_j$

$$\Rightarrow \quad M\dot{\xi}(t) + A\xi(t) + C\xi(t) = b$$

where
$$m_{ij} = \int_\Omega \varphi_i \cdot \varphi_j \, dx$$
$$a_{i,j} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx$$
$$c_{i,j} = \int_\Omega c\varphi_i \cdot \varphi_j \, dx$$
$$b_j = \int_\Omega f \cdot \varphi_j \, dx$$

Exe 5.4.4

Forward euler $t_{n-1} - t_{n-2} = k$ $n = 1, \ldots$

let $\xi^n \approx \xi(t_n)$

$$M \frac{\xi^n - \xi^{n-1}}{k} + A\xi^{n-1} + C\xi^{n-1} = b^{n-1},$$

where $b_j^{n-1} = \int_\Omega f(t^{n-1}) \varphi_j \, dx$

$$\therefore \quad M\xi^n = (M - k(A + C))\xi^{n-1} + kb^{n-1}, \quad \xi^0 = u_0$$

$n = 1, \ldots,$

Exe 5.4.5

Consider
$$\ddot{u} - \Delta u = f, \quad x \in \Omega, \ t > 0$$
$$u = 0, \quad x \in \partial\Omega$$
$$u(\cdot,0) = u_0, \quad x \in \Omega$$
$$\dot{u}(\cdot,0) = v_0, \quad x \in \Omega$$

Choose f, u_0, v_0 s.t $u = \sin\pi x \sin\pi y \sin\pi t$

Sol

$$\ddot{u} - \Delta u = (-\pi^2 + 2\pi^2)u = \underbrace{\pi^2 \sin\pi x \sin\pi y \sin\pi t}_{f(x,y,t)}$$

$$u(\cdot,0) = 0 = u_0$$

$$\dot{u}(\cdot,0) = \pi\cos\pi\cdot 0 \sin\pi x \sin\pi y = \underbrace{\pi \sin\pi x \sin\pi y}_{v_0}$$

Exe 5.4.6

Discretize using Backward Euler + FEM

Sol,

Formulate Weak form: Find $u \in V_0$ s.t
$$(\ddot{u}, v) + (\nabla u, \nabla v) = (f, v) \quad \forall v \in V_0$$

FEM: Find $U \in V_{h,0}$ s.t
$$(\ddot{U}, v) + (\nabla U, \nabla v) = (f, v) \quad \forall v \in V_{0,h}$$

$$\Rightarrow M\ddot{\xi} + A\xi = b \quad \text{where } \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

$$U = \sum_{i=1}^{n} \xi_i \varphi_i, \quad m_{ij} = \int_\Omega \varphi_i \cdot \varphi_j \, dx$$

$$a_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx, \quad b_j = \int_\Omega f \varphi_j \, dx$$

We have a second order system

let $M\eta = M\dot{\xi}$ $\Rightarrow$ $M\dot{\eta} + A\xi = b$

$M\eta = M\dot{\xi}$

$$M\frac{\xi^n - \xi^{n-1}}{k} = M\eta^n$$

$$M\frac{\eta^n - \eta^{n-1}}{k} = -A\xi^n + b^n$$

$$\Rightarrow \begin{bmatrix} M & -kM \\ -kA & M \end{bmatrix}\begin{bmatrix} \xi^n \\ \eta^n \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \xi^{n-1} \\ \eta^{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ k\cdot b^n \end{bmatrix}$$

Exe 5.4.7

Consider $-i\dot{u} - \varepsilon\Delta u = 0 \quad x\in\Omega$

$u = 0 \quad x\in\partial\Omega$

$u(\cdot,0) = u_0 + iv_0, \quad x\in\Omega$

Sol

$$-i(\dot{v} + i\dot{w}) - \varepsilon\Delta(v + iw) = 0$$

$1:\ \dot{w} - \varepsilon\Delta v = 0$ $\qquad u = 0 \Rightarrow$ $v = 0 \quad \partial\Omega$

$i:\ -\dot{v} - \varepsilon\Delta w = 0$ $\qquad w = 0 \quad \partial\Omega$

FEM: $\quad M\dot{\eta} + \varepsilon A\xi = 0$ $\qquad v = \sum \xi_i\varphi_i$

$-M\dot{\xi} + \varepsilon A\eta = 0$ $\qquad w = \sum \eta_i\varphi_i$

$m_{ij} = \int_\Omega \varphi_i\cdot\varphi_j\,dx$

$a_{ij} = \int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\,dx$

$$M\frac{\xi_n - \xi_{n-1}}{k} - \varepsilon\frac{A\eta_n + A\eta_{n-1}}{2} = 0$$

$$M\frac{\eta_n - \eta_{n-1}}{k} + \varepsilon\frac{A\xi_n + A\xi_{n-1}}{2} = 0$$

$$M\xi^n - \varepsilon\tfrac{k}{2}A\eta^n = M\xi^{n-1} + \varepsilon\tfrac{k}{2}A\eta^{n-1}$$

$$M\eta^n + \varepsilon\tfrac{k}{2}A\xi^n = M\eta^{n-1} - \varepsilon\tfrac{k}{2}A\xi^{n-1}$$

On matrix form

$$\begin{bmatrix} M & -\varepsilon\frac{k}{2}A \\ \varepsilon\frac{k}{2}A & M \end{bmatrix}\begin{bmatrix} \xi^n \\ \eta^n \end{bmatrix} = \begin{bmatrix} M & \varepsilon\frac{k}{2}A \\ -\varepsilon\frac{k}{2}A & M \end{bmatrix}\begin{bmatrix} \xi^{n-1} \\ \eta^{n-1} \end{bmatrix}$$

$\xi^0 = v^0, \quad \eta^0 = w^0.$

Exe 5.4.8

Show $\frac{\partial}{\partial t}\|u\|^2 = 0$.

Sol.

Multiply with $\bar{u} = v - iw$ and $\int_\Omega$

$$\int_\Omega -i\dot{u}\,\bar{u}\,dx = \varepsilon\int_\Omega \Delta u\cdot\bar{u}\,dx \overset{\text{Green}}{=} -\varepsilon\int_\Omega \nabla u\cdot\nabla\bar{u}\,dx =$$

$$= -\varepsilon\int_\Omega \nabla u\cdot\overline{\nabla u}\,dx = -\varepsilon\|\nabla u\|^2 \in \mathbb{R}$$

$$\int_\Omega -i\dot{u}\,\bar{u}\,dx = \int_\Omega -i(\dot{v}+i\dot{w})(v-iw)\,dx = \int_\Omega(-i\dot{v}+\dot{w})(v-iw)\,dx$$

$$= \int_\Omega -i\dot{v}\cdot v - \dot{v}\cdot w + \dot{w}\cdot v - i\dot{w}\cdot w\,dx =$$

$$= -\frac{1}{2}i\frac{\partial}{\partial t}\left(\int_\Omega v^2dx + \int_\Omega w^2dx\right) + \int_\Omega(\dot{w}\cdot v - \dot{v}\cdot w)\,dx$$

$$\Rightarrow \frac{\partial}{\partial t}\int_\Omega v^2 + w^2\,dx = 0 \quad (-\varepsilon\|\nabla u\|^2 \in \mathbb{R})$$

$$\frac{\partial}{\partial t}\|u\|^2 = \frac{\partial}{\partial t}\int_\Omega |v+iw|^2dx = \frac{\partial}{\partial t}\int_\Omega v^2+w^2\,dx = 0 \quad \blacksquare$$