PROBLEMS CHAPTER 5

Exercise 5.4.1 Consider the equation,

$$\dot{u} - \Delta u + cu = f, \quad x \in \Omega, \quad t > 0, \tag{1}$$
$$u = 0, \quad x \in \partial\Omega,$$
$$u(\cdot, 0) = u_0, \quad x \in \Omega,$$

where $\Omega = [0,1] \times [0,1]$ is the unit square and $c \geq 0$. Choose f and u_0 so that $u(x,y,t) = e^{-t} sin(\pi x) sin(\pi y)$ solves equation (1) if $c = 0$.

Exercise 5.4.2 Let $f = 0$ in equation (1). Show that,

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)},$$

for all $t \geq 0$. *Hint:* multiply equation (1) by u and integrate in space. Note that $2v\dot{v} = \frac{\partial}{\partial t}(v^2)$.

Exercise 5.4.3 Derive the weak form of equation (1). Discretize the problem in space using the finite element method by choosing an appropriate discrete functions space. Write the results a linear system of ordinary differential equations.

Exercise 5.4.4 Discretize in time using forward Euler. Let the time interval $[0,T]$ be divided into subdomains of equal length k. Derive the resulting linear system of equations without computing any entries.

Exercise 5.4.5 Consider the equation,

$$\ddot{u} - \Delta u = f, \quad x \in \Omega, \quad t > 0, \tag{2}$$
$$u = 0, \quad x \in \partial\Omega,$$
$$u(\cdot, 0) = u_0, \quad x \in \Omega,$$
$$\dot{u}(\cdot, 0) = v_0, \quad x \in \Omega,$$

where $\Omega = [0,1] \times [0,1]$ is the unit square. Choose f, u_0, and v_0 so that $u(x,y,t) = sin(\pi x) sin(\pi y) sin(\pi t)$ solves equation (2).

Exercise 5.4.6 Choose a suitable function space and construct the weak form of equation (2). Discretize in space using the finite element method and formulate a system of first order linear ordinary differential equations. Discretize in time using backward Euler, with time step k, and state the resulting algebraic system.

Exercise 5.4.7 Consider the Schrödinger equation for a particle in a box,

$$
\begin{aligned}
-i\dot{u} - \epsilon\Delta u &= 0, \quad x \in \Omega, \quad t > 0, \\
u &= 0, \quad x \in \partial\Omega, \\
u(\cdot, 0) &= v_0 + iw_0, \quad x \in \Omega,
\end{aligned} \tag{3}
$$

where $\Omega = [0,1] \times [0,1]$ is the unit square, i is the imaginary unit, $\epsilon = \frac{\hbar}{2m}$, and $u = v + iw$, where v, w are real valued functions. Discretize in space using the finite element method and in time using Crank-Nicholson and present the time stepping algorithm on matrix form. *Hint:* Let $u = v + iw$ in (3), both equation and initial condition, and identify all imaginary terms and then all real terms. Both should sum up to 0 individually. This will lead to a system similar to system one gets when discretizing the wave equation.

Exercise 5.4.8 Show that the quantity $\|u\|^2 = \int_\Omega u\bar{u}\,dx$, where $\bar{u} = v - iw$, in equation (3) is constant in time. *Hint:* Multiply equation (3) with $\bar{u}$ and integrate over Ω. Use that $\frac{\partial}{\partial t}(v^2) = 2v\dot{v}$. Again identify real and imaginary parts.