UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2009-03-13

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let V_h be the space of continuous piecewise linear functions on a partition $0 = x_0 < x_1 < \cdots < x_N = 1$ of the interval $[0, 1]$. Further let $f(x) = 5x^2 - 3x$.

a) Let $N = 1$, that is $x_0 = 0$ and $x_1 = 1$. Compute the interpolant $\pi_h f$ and the $L^2([0, 1])$ projection $P_h f$. [2p.]

b) Again let $N = 1$. Compute approximations to $\int_0^1 f(x)\,dx$ using the midpoint rule and the trapezoidal rule. Also compute the errors in these approximations compared to the exact solution. [2p.]

c) Now let $N = 10$ and assume that the nodes are equidistributed, that is $x_i - x_{i-1} = 0.1$ for all $i = 1, \ldots, N$. Compute element $m_{0,1} = \int_0^1 \phi_0 \phi_1\,dx$ in the mass matrix. Here $\phi_i \in V_h$ are the hat functions equal to one in node i and zero in all other nodes. [2p.]

d) Show that the mass matrix is symmetric and positive definite, that is $m_{i,j} = m_{j,i}$ and $v^T M v \geq 0$ for all $v \in \mathbf{R}^{N+1}$ and $v^T M v = 0$ if and only if $v = 0$. [2p.]

Problem 2

Consider the problem, find $u(x, y)$ such that,

$$-\Delta u = f, \quad x \in \Omega \qquad (1)$$

$$u = 0, \quad x \in \partial\Omega, \qquad (2)$$

where $\Omega = [0, 1] \times [0, 1]$ is the unit square with boundary $\partial\Omega$.

a) Determine f so that $u = x(1-x)y(1-y)$ solves equation (1). Check that the boundary condition (2) is fulfilled. [2p.]

b) Let $V_0 = \{v : \|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} < \infty,\ v = 0 \text{ on } \partial\Omega\}$. Derive the weak form of equation (1). Let f be arbitrary. [2p.]

c) Let $V_{h,0}$ be the finite element space of piecewise linear continuous functions, on a triangulation $\mathcal{K}$ of Ω, which are zero on the boundary. Derive the finite element method. You do not have to write the method on matrix form. [2p.]

d) Show that $\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq C_2 \|f\|^2_{L^2(\Omega)}$ for some constant C_2. You can assume that the Poincare-Friedrich inequality holds, $\|u\|_{L^2(\Omega)} \leq C_1 \|\nabla u\|_{L^2(\Omega)}$. [2p.]

Problem 3

Consider the problem, find $u(x,y,t)$ such that

$$\begin{aligned} \dot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \leq T \\ u &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ u &= u_0, \quad x \in \Omega, \quad t = 0, \end{aligned} \tag{3}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) For each fixed $t > 0$, formulate the weak form of equation (3) by multiplying with a test function in a function space V_0 and integrating over Ω. [2p.]

b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. [2p.]

c) Discretize in time by dividing the time interval $[0,T]$ into N subintervals of equal length. Formulate the Backward Euler method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [2p.]

d) Show that $\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}$ for all $t > 0$. [2p.]

Problem 4

Consider the problem, find $u = u(x)$ such that,

$$\begin{aligned} -u''(x) &= f, \quad x \in [0,1], \\ u(0) &= u(1) = 0, \end{aligned}$$

where f is a given function. Let $0 = x_0 < x_1 < \cdots < x_N = 1$. We denote each subinterval $I_i = [x_{i-1}, x_i]$ and let $h_i = x_i - x_{i-1}$, for all $i = 1, \ldots, N$. We construct a finite element space $V_{h,0} = \{v \in C([0,1]) : v \text{ linear on } I_i, \text{ for all } i = 1, \ldots, N, v(0) = v(1) = 0\}$ and let $U \in V_{h,0}$ be the finite element approximation of u.

a) Derive the a priori error bound for $\|(u-U)'\|_{L^2([0,1])}$. You can assume that $\|(u - \pi_h u)'\|_{L^2(I_i)} \leq Ch_i\|u''\|_{L^2(I_i)}$ for all $i = 1, \ldots, N$, where π_h is the interpolant onto $V_{h,0}$. [2p.]

b) What does the a priori estimate tell us about the convergence of the method? [2p.]

c) Derive the a posteriori error bound for $\|(u-U)'\|_{L^2([0,1])}$. You can assume that $\|(u-U) - \pi_h(u-U)\|_{L^2(I_i)} \leq Ch_i\|(u-U)'\|_{L^2(I_i)}$ for all $i = 1, \ldots, N$. [2p.]

d) Describe how the a posteriori error bound can be used in an adaptive algorithm to improve the solution U by adding more nodes in the mesh. [2p.]

Problem 5

Consider the problem, find $u(x,y)$ such that,

$$\begin{aligned} -\Delta u + u &= f, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$. Let $\langle v, w \rangle = (\nabla v, \nabla w) + (v, w)$ be a scalar product with corresponding norm $\|v\|^2_{H^1(\Omega)} = \langle v, v \rangle = \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}$. We use the notation $(v,w) = \int_\Omega vw\,dx$.

a) Show that $\langle \cdot, \cdot \rangle$ is a scalar product, that is show that (i) $\langle v, w \rangle = \langle w, v \rangle$, (ii) $\langle \alpha v + \beta w, z \rangle = \alpha \langle v, z \rangle + \beta \langle w, z \rangle$ for $\alpha, \beta \in \mathbf{R}$, and that (iii) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$. You can use that $(\cdot, \cdot)$ is a scalar product, that is it fulfills (i-iii). [2p.]

b) Let $V_{h,0}$ be the space of continuous piecewise linear functions on a mesh $\mathcal{K}$, of Ω, that are equal to zero on the boundary $\partial\Omega$. Further let $U \in V_{h,0}$ be the finite element approximation of u. Show that the error $u - U$ is orthogonal to $V_{h,0}$ in the $\langle \cdot, \cdot \rangle$ scalar product, that is show $\langle u - U, v \rangle = 0$, for all $v \in V_{h,0}$. [3p.]

c) Show that U is the best approximation to u within the space $V_{h,0}$ if the distance is measured in the norm $\| \cdot \|_{H^1(\Omega)}$, that is show that $\|u - U\|_{H^1(\Omega)} \leq \|u - v\|_{H^1(\Omega)}$ for all $v \in V_{h,0}$. [3p.]

Good luck!
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