UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2009-03-13

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let V_h be the space of continuous piecewise linear functions on a partition $0 = x_0 < x_1 < \cdots < x_N = 1$ of the interval $[0, 1]$. Further let $f(x) = 5x^2 - 3x$.

a) Let $N = 1$, that is $x_0 = 0$ and $x_1 = 1$. Compute the interpolant $\pi_h f$ and the $L^2([0, 1])$ projection $P_h f$. [2p.]
Solution: *Let $\phi_0 = 1 - x$ and $\phi_1 = x$. We have $\pi_h f = f(x_0)\phi_0 + f(x_1)\phi_1 = 2x$.*

b) Again let $N = 1$. Compute approximations to $\int_0^1 f(x)\,dx$ using the midpoint rule and the trapezoidal rule. Also compute the errors in these approximations compared to the exact solution. [2p.]
Solution: *First we compute the exact integral, $I = \int_0^1 f(x)\,dx = 1/6$. Midpoint: $I \approx (1-0)\cdot f(1/2) = -1/4$, with error $|1/6 - (-1/4)| = 5/12$. Trapezoidal: $I \approx (1-0)\cdot (f(0) + f(1))/2 = 1$, with error $|1/6 - 1| = 5/6$.*

c) Now let $N = 10$ and assume that the nodes are equidistributed, that is $x_i - x_{i-1} = 0.1$ for all $i = 1, \ldots, N$. Compute element $m_{0,1} = \int_0^1 \phi_0\phi_1\,dx$ in the mass matrix. Here $\phi_i \in V_h$ are the hat functions equal to one in node i and zero in all other nodes. [2p.]
Solution: *The basis functions ϕ_0 and ϕ_1 only overlap on $[0, 0.1]$. On this interval $\phi_0 = 1 - 10x$ and $\phi_1 = 10x$. We get $\int_0^{0.1} \phi_0\phi_1\,dx = \int_0^{0.1} 10x(1-10x)\,dx = 1/60$.*

d) Show that the mass matrix is symmetric and positive definite, that is $m_{i,j} = m_{j,i}$ and $v^T M v \geq 0$ for all $v \in \mathbf{R}^{N+1}$ and $v^T M v = 0$ if and only if $v = 0$. [2p.]
Solution: *See Thm 3.4 in the December 18, 2009 edition of the lecture notes. The argument in 2D is also valid in 1D.*

Problem 2

Consider the problem, find $u(x, y)$ such that,

$$-\Delta u = f, \quad x \in \Omega \tag{1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{2}$$

where $\Omega = [0, 1] \times [0, 1]$ is the unit square with boundary $\partial\Omega$.

a) Determine f so that $u = x(1-x)y(1-y)$ solves equation (1). Check that the boundary condition (2) is fulfilled. [2p.]
Solution: *We first check the boundary conditions: u(0,y)=u(1,y)=u(x,0)=u(x,1)=0. Then we plug $u = x(1-x)y(1-y)$ into the equation. We get, $-\triangle x(1-x)y(1-y) = 2y(1-y) + 2x(1-x) = f(x, y)$.*

b) Let $V_0 = \{v : \|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} < \infty,\ v = 0 \text{ on } \partial\Omega\}$. Derive the weak form of equation (1). Let f be arbitrary. [2p.]
Solution: *Find $u \in V_0$ such that, $(f, v) = (-\triangle u, v) =$ Green's formula $= (\nabla u, \nabla v) - (\partial_n u, v)_{\partial\Omega} =$ boundary conditions on $v = (\nabla u, \nabla v)$, for all $v \in V_0$. Here $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ scalar product and $(\cdot,\cdot)_{\partial\Omega}$ denotes the $L^2(\partial\Omega)$ scalar product.*

c) Let $V_{h,0}$ be the finite element space of piecewise linear continuous functions, on a triangulation $\mathcal{K}$ of Ω, which are zero on the boundary. Derive the finite element method. You do not have to write the method on matrix form. [2p.]
Solution: *Find $u_h \in V_{h,0}$ such that, $(\nabla u_h, \nabla v) = (f, v)$ for all $v \in V_{h,0}$.*

d) Show that $\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq C_2\|f\|^2_{L^2(\Omega)}$ for some constant C_2. You can assume that the Poincare-Friedrich inequality holds, $\|u\|_{L^2(\Omega)} \leq C_1\|\nabla u\|_{L^2(\Omega)}$. [2p.]
Solution: *Start with weak form and let $v = u$. We get $\|\nabla u\|^2_{L^2(\Omega)} = (f, u) =$ Cauchy-Schwarz $\leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq$ Poincare-Friedrich $\leq C_1\|f\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega}$. Assume $\nabla u \neq 0$, otherwise the statement holds trivially because of the PF inequality. Divide by $\|\nabla u\|_{L^2(\Omega)}$ and use PF again to get, $\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C_1 + 1)\|\nabla u\|^2_{L^2(\Omega)} \leq C_1(C_1+1)\|f\|_{L^2(\Omega)} := C_2\|f\|_{L^2(\Omega)}$.*

Problem 3

Consider the problem, find $u(x, y, t)$ such that

$$\begin{aligned} \dot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \leq T \\ u &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ u &= u_0, \quad x \in \Omega, \quad t = 0, \end{aligned} \tag{3}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) For each fixed $t > 0$, formulate the weak form of equation (3) by multiplying with a test function in a function space V_0 and integrating over Ω. [2p.]
Solution: *Let $u(0) = u_0$. For each $t > 0$ find $u(t) \in V_0$ such that, $(\dot{u}, v) + (\nabla u, \nabla v) = 0$ for all $v \in V_0$. We have used Green's formula to more the gradient to the test function.*

b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. [2p.]
Solution: *Let $V_{h,0} = \{v \in C_0(\Omega) : v|_K$ is linear for all triangles K in the mesh $\mathcal{K}\}$, and span$(\{\phi_i\}_{i=1}^N) = V_{h,0}$. The finite element method reads: With initial condition $U(0) = \pi_h u_0$, for each $t > 0$, find $U \in V_{h,0}$ such that, $(\dot{U}, v) + (\nabla U, \nabla v) = 0$ for all $v \in V_{h,0}$. We plug $U = \sum \xi_i(t)\phi_i$ into the finite element method and note that $\{\phi_j\}_{j=1}^N$ is a basis for $V_{h,0}$ i.e. we only need to test with the basis functions. We get $\sum_{i=1}^N \big(\dot{\xi}_i(t)(\phi_i, \phi_j) + \xi_i(t)(\nabla\phi_i, \nabla\phi_j)\big) = 0$, for all $j = 1, \ldots, N$. We let $m_{i,j} = (\phi_i, \phi_j)$ and $a_{i,j} = (\nabla\phi_i, \nabla\phi_j)$ for all $i, j = 1, \ldots, N$. We get the following system of ordinary differential equations, $\xi_i(0) = u_0(x_i)$, $M\dot{\xi} + A\xi = 0$, with the obvious definition of A and M and with x_i as the nodal coordinates.*

c) Discretize in time by dividing the time interval $[0, T]$ into N subintervals of equal length. Formulate the Backward Euler method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [2p.]
Solution: *Let $k = T/N$ be the time step. Backward Euler: $M\xi^{n+1} = M\xi^n - kA\xi^{n+1} = 0$, for all $n = 0, \ldots, N-1$, with $\xi^0 = \xi(0)$. In each iteration we need to solve $(M + kA)\xi^{n+1} = M\xi^n$.*

d) Show that $\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}$ for all $t > 0$. [2p.]
Solution: *We consider the weak form with $v = u$ and note that $2u\dot{u} = \frac{\partial}{\partial t}(u^2)$. We*

get, $\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} = 0$. *We conclude that* $\|u\|_{L^2(\Omega)}$ *is decreasing since the time derivative is negative. Therefore it can never be bigger then its initial value* $\|u_0\|_{L^2(\Omega)}$.

Problem 4

Consider the problem, find $u = u(x)$ such that,

$$-u''(x) = f, \quad x \in [0, 1],$$
$$u(0) = u(1) = 0,$$

where f is a given function. Let $0 = x_0 < x_1 < \cdots < x_N = 1$. We denote each subinterval $I_i = [x_{i-1}, x_i]$ and let $h_i = x_i - x_{i-1}$, for all $i = 1, \ldots, N$. We construct a finite element space $V_{h,0} = \{v \in C([0, 1]) : v \text{ linear on } I_i, \text{ for all } i = 1, \ldots, N, v(0) = v(1) = 0\}$ and let $U \in V_{h,0}$ be the finite element approximation of u.

a) Derive the a priori error bound for $\|(u - U)'\|_{L^2([0,1])}$. You can assume that $\|(u - \pi_h u)'\|_{L^2(I_i)} \leq Ch_i\|u''\|_{L^2(I_i)}$ for all $i = 1, \ldots, N$, where π_h is the interpolant onto $V_{h,0}$. [2p.]
Solution: *See Thm 2.1-2.3 in the December 18, 2009, edition of the lecture notes.*

b) What does the a priori estimate tell us about the convergence of the method? [2p.]
Solution: *The derivative of the error in* L^2 *norm is bounded by the maximum mesh size and therefore decreases to zero as the mesh size decreases to zero. Poincare-Friedrich inequality also makes the* L^2 *norm of the error decrease to zero with the mesh size, i.e. the method is convergent. Furthermore, the bigger the second derivative of the exact solution the more difficult it is to approximate.*

c) Derive the a posteriori error bound for $\|(u - U)'\|_{L^2([0,1])}$. You can assume that $\|(u - U) - \pi_h(u - U)\|_{L^2(I_i)} \leq Ch_i\|(u - U)'\|_{L^2(I_i)}$ for all $i = 1, \ldots, N$. [2p.]
Solution: *See Prop 2.1 in the December 18, 2009, edition of the lecture notes.*

d) Describe how the a posteriori error bound can be used in an adaptive algorithm to improve the solution U by adding more nodes in the mesh. [2p.]
Solution: *See Algorithm 6 in the December 18, 2009, edition of the lecture notes. Equation 2.82 is an example of a criteria for choosing which intervals to refine.*

Problem 5

Consider the problem, find $u(x, y)$ such that,

$$-\Delta u + u = f, \quad x \in \Omega$$
$$u = 0, \quad x \in \partial\Omega,$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$. Let $\langle v, w\rangle = (\nabla v, \nabla w) + (v, w)$ be a scalar product with corresponding norm $\|v\|^2_{H^1(\Omega)} = \langle v, v\rangle = \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}$. We use the notation $(v, w) = \int_\Omega vw\,dx$.

a) Show that $\langle\cdot,\cdot\rangle$ is a scalar product, that is show that (i) $\langle v, w\rangle = \langle w, v\rangle$, (ii) $\langle \alpha v + \beta w, z\rangle = \alpha\langle v, z\rangle + \beta\langle w, z\rangle$ for $\alpha, \beta \in \mathbf{R}$, and that (iii) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$. You can use that $(\cdot,\cdot)$ is a scalar product, that is it fulfills (i-iii). [2p.]
Solution: *(i)* $\langle v, w\rangle = (\nabla v, \nabla w) + (v, w) = (\nabla w, \nabla v) + (w, v) = \langle w, v\rangle$, *i.e. it follows from the symmetry of the ordinary product inside the integral. (ii)* $\langle \alpha v + \beta w, z\rangle = (\nabla(\alpha v + \beta w), \nabla z) + (\alpha v + \beta w, z) = \alpha((\nabla v, \nabla z) + (v, z)) + \beta((\nabla w, \nabla z) + (w, z)) = \alpha\langle v, z\rangle + \beta\langle w, z\rangle$. *(iii)* $\langle v, v\rangle = \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \geq 0$. *We only have equality if* $\|v\|_{L^2(\Omega)} = 0$ *but then* $v = 0$ *since* $\|\cdot\|_{L^2(\Omega)}$ *is a norm. If* $v = 0$ *then* $\langle v, v\rangle = 0$ *trivially.*

b) Let $V_{h,0}$ be the space of continuous piecewise linear functions on a mesh $\mathcal{K}$, of Ω, that are equal to zero on the boundary $\partial\Omega$. Further let $U \in V_{h,0}$ be the finite element approximation of u. Show that the error $u - U$ is orthogonal to $V_{h,0}$ in the $\langle \cdot, \cdot \rangle$ scalar product, that is show $\langle u - U, v \rangle = 0$, for all $v \in V_{h,0}$. [3p.]
Solution: *The weak form reads: find* $u \in V_0$ *such that,* $(\nabla u, \nabla v) + (u, v) = (f, v)$ *for all* $v \in V_0$. *The finite element method reads: find* $u_h \in V_{h,0}$ *such that,* $(\nabla U, \nabla v) + (U, v) = (f, v)$ *for all* $v \in V_{h,0} \subset V_0$. *Subtracting the second from the first gives* $\langle u - U, v \rangle = (\nabla(u - U), \nabla v) + (u - U, v) = (f, v) - (f, v) = 0$ *for all* $v \in V_{h,0}$.

c) Show that U is the best approximation to u within the space $V_{h,0}$ if the distance is measured in the norm $\| \cdot \|_{H^1(\Omega)}$, that is show that $\|u - U\|_{H^1(\Omega)} \leq \|u - v\|_{H^1(\Omega)}$ for all $v \in V_{h,0}$. [3p.]
Solution: *We start with the left hand side and use the Galerkin Orthogonality from (b). We get,* $\|u - U\|^2_{H^1(\Omega)} = \langle u - U, u - U \rangle = \langle u - U, u - v \rangle = (\nabla(u - U), \nabla(u - v)) + (u - U, u - v) \leq \|\nabla(u - U)\|_{L^2(\Omega)} \|\nabla(u - v)\|_{L^2(\Omega)} + \|u - U\|_{L^2(\Omega)} \|u - v\|_{L^2(\Omega)}$, *for all* $v \in V_{h,0}$. *Next we use the observation* $2ab \leq a^2 + b^2$ *for real numbers* a *and* b. *We get,* $\|u - U\|^2_{H^1(\Omega)} \leq \frac{1}{2}\|\nabla(u - U)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u - v)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - U\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - v\|^2_{L^2(\Omega)} = \frac{1}{2}\|u - U\|^2_{H^1(\Omega)} + \frac{1}{2}\|u - v\|^2_{H^1(\Omega)}$. *We can now subtract* $\frac{1}{2}\|u - U\|^2_{H^1(\Omega)}$ *from both sides and the statement follows.*

Good luck!
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