UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2009-06-11

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let $\Omega = [0,1] \times [0,0.5]$. Consider the triangulation in Figure 1. Let the nodes be numbered in the following way $N_1 = (0,0)$, $N_2 = (0.5,0)$, $N_3 = (1,0)$, $N_4 = (1,0.5)$, $N_5 = (0.5,0.5)$, and $N_6 = (0,0.5)$. Let $\{\varphi_i\}_{i=1}^6$ be piecewise linear continuous basis functions fulfilling $\varphi_i(N_j) = 1$ when $i = j$ and $\varphi_i(N_j) = 0$ otherwise. Further let $f(x,y) = x^2$.

a) Derive the p and t matrix describing the location of the nodes and the triangles of the mesh. Note that you can choose any numbering of the triangles. [2p.]

b) Give an analytical expression for φ_1. [2p.]

c) Compute the nodal interpolant πf as a linear combination of the basis functions $\{\varphi_i\}_{i=1}^6$. [2p.]

d) When computing the L^2-projection Pf the mass matrix with entries $m_{ij} = \int_\Omega \varphi_i \varphi_j \, dx$ needs to be derived. Compute m_{24}. [2p.]

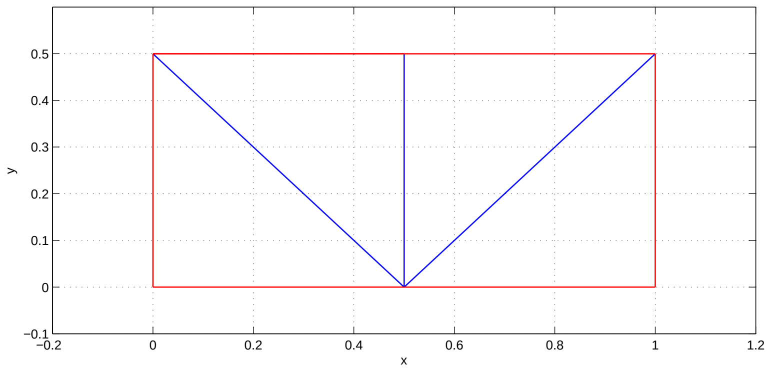

Figure 1: Mesh in 2D.

Problem 2

Consider the problem, find $u(x)$ such that,

$$-(a(x)u'(x))' = f, \quad x \in (0,1) \tag{1}$$
$$a(0)u'(0) = \kappa u(0),$$
$$-a(1)u'(1) = \kappa u(1),$$

where $a(x) \geq a_0 > 0$, $\kappa \geq 0$, and f are given functions.

a) Derive the weak form of equation (1). [2p.]

b) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a discretization of $(0,1)$. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. Do *not* compute matrix and vector entries. [2p.]

c) In the stiffness matrix, terms of the following form appear $\int_0^1 a(x)\varphi_i(x)'\varphi_j(x)'\,dx$. Present two different quadrature rules for approximating $\int_0^1 a(x)\varphi_0(x)'\varphi_0(x)'\,dx$, where $\varphi_0(x)$ is the basis function corresponding to node x_0. Note that a is arbitrary positive. [2p.]

d) Show that,

$$\int_0^1 a(x)|u'(x)|^2\,dx + \kappa|u(1)|^2 + \kappa|u(0)|^2 \le C\int_0^1 |f(x)|^2\,dx,$$

for some constant $C > 0$. You can assume that $\int_0^1 |u(x)|^2\,dx \le C'\int_0^1 |u'(x)|^2\,dx$ for some constant $C' > 0$. [2p.]

Problem 3

Consider the problem, find $u(x,y,t)$ such that

$$\begin{aligned} \ddot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \le T \\ u &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ u &= u_0, \quad x \in \Omega, \quad t = 0, \\ \dot{u} &= v_0, \quad x \in \Omega, \quad t = 0, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) Write equation (2) as a system of two first order (in time) equations. For each fixed $t > 0$, formulate the weak form. [3p.]

b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. Let $V_{h,0} \subset V_0 = \{v : \|v\|_{L^2(\Omega)} < \infty, \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$ be the discrete approximation space. [2p.]

c) Discretize in time by dividing the time interval $[0,T]$ into N subintervals of equal length. Formulate the Crank-Nicholson method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [3p.]

Problem 4

Consider the weak form of the Dirichlet problem on a domain $\Omega \subset \mathbf{R}^2$, find $u \in V_0$ such that,

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega fv\,dx, \quad \text{for all } v \in V_0, \tag{3}$$

where $V_0 = \{v : \|v\|_{L^2(\Omega)} < \infty, \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$. Further let $U \in V_{h,0}$ be the finite element approximation of u, where $V_{h,0} \subset V_0$ is the space of continuous piecewise linear functions on a triangulation $\mathcal{K} = \{K\}$ of Ω.

a) Show that,

$$\int_\Omega \nabla(u - U)\cdot\nabla v\,dx = 0, \quad \text{for all } v \in V_{h,0}.$$

[3p.]

b) Show that,

$$\int_\Omega |\nabla(u-U)|^2\,dx \leq \int_\Omega |\nabla(u-v)|^2\,dx, \quad \text{for all } v \in V_{h,0}.$$

[3p.]

c) Show that

$$\int_\Omega |\nabla(u-U)|^2\,dx \leq \sum_{K\in\mathcal{K}} Ch_K^2\|D^2u\|^2_{L^2(K)},$$

where $h_K = \text{diam}(K)$. The following interpolation estimate can be assumed to be true,

$$\|\nabla(u-\pi u)\|^2_{L^2(K)} \leq Ch_K^2\|D^2u\|^2_{L^2(K)}.$$

[2p.]

Problem 5

Consider the problem, find $u = u(x)$ such that,

$$-u''(x) + cu(x) = f(x), \quad x \in [0,1], \tag{4}$$

$$u(0) = u(1) = 0, \tag{5}$$

where $f(x)$ is a given function and $c \geq 0$ is a given constant. Let $0 = x_0 < x_1 < \cdots < x_N = 1$. We denote each subinterval $I_i = [x_{i-1}, x_i]$ and let $h_i = x_i - x_{i-1}$, for all $i = 1,\ldots,N$. We construct a finite element space $V_{h,0} = \{v \in C([0,1]) : v \text{ linear on } I_i, \text{ for all } i = 1,\ldots,N, v(0) = v(1) = 0\}$.

a) Derive the weak form of equation (4) and the finite element method using the space $V_{h,0}$. [3p.]

b) Derive the a posteriori error bound for $\left(\|(u-U)'\|^2_{L^2([0,1])} + c\|u-U\|^2_{L^2([0,1])}\right)^{1/2}$. You can assume that $\|(u-U)-\pi_h(u-U)\|_{L^2(I_i)} \leq Ch_i\|(u-U)'\|_{L^2(I_i)}$ for all $i = 1,\ldots,N$, where U is the finite element approximation. [3p.]

c) Describe how the a posteriori error bound can be used in an adaptive algorithm to improve the solution U by adding more nodes in the mesh. [2p.]

Good luck!
Axel Målqvist