UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2009-06-11

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let $\Omega = [0,1]\times[0,0.5]$. Consider the triangulation in Figure 1. Let the nodes be numbered in the following way $N_1 = (0,0)$, $N_2 = (0.5,0)$, $N_3 = (1,0)$, $N_4 = (1,0.5)$, $N_5 = (0.5,0.5)$, and $N_6 = (0,0.5)$. Let $\{\varphi_i\}_{i=1}^6$ be piecewise linear continuous basis functions fulfilling $\varphi_i(N_j) = 1$ when $i = j$ and $\varphi_i(N_j) = 0$ otherwise. Further let $f(x,y) = x^2$.

a) Derive the p and t matrix describing the location of the nodes and the triangles of the mesh. Note that you can choose any numbering of the triangles. [2p.]
Solution. For example,

$$p = \begin{bmatrix} 0 & 0.5 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \end{bmatrix} \quad t = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 3 & 4 \\ 6 & 6 & 4 & 5 \end{bmatrix}.$$

b) Give an analytical expression for φ_1. [2p.]
Solution. The other two basis functions on that non-zero on this triangle are easier to directly write an expression for, $\phi_2 = 2x$ and $\phi_6 = 2y$. Since they add up to one we get $\phi_1 = 1 - 2x - 2y$ on the lower left triangle and zero in the rest of the domain.

c) Compute the nodal interpolant πf as a linear combination of the basis functions $\{\varphi_i\}_{i=1}^6$. [2p.]
Solution. $\pi f = \sum_{i=1}^6 f(x_i)\phi_i = 0.25\phi_2 + \phi_3 + \phi_4 + 0.5\phi_5$.

d) When computing the L^2-projection Pf the mass matrix with entries $m_{ij} = \int_\Omega \varphi_i\varphi_j\,dx$ needs to be derived. Compute m_{24}. [2p.]
Solution. The integrand in m_{24} is non-zero on two triangle. We use equation 3.65 in the book to get, $m_{24} = 2\frac{1}{8}\frac{2\cdot 1!1!}{(1+1+2)!} = \frac{1}{48}$.

Problem 2

Consider the problem, find $u(x)$ such that,

$$\begin{aligned} -(a(x)u'(x))' &= f, \quad x \in (0,1) \\ a(0)u'(0) &= \kappa u(0), \\ -a(1)u'(1) &= \kappa u(1), \end{aligned} \tag{1}$$

where $a(x) \geq a_0 > 0$, $\kappa \geq 0$, and f are given functions.

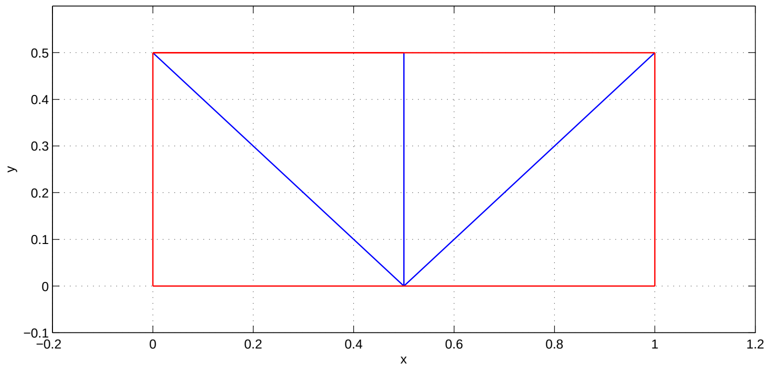

Figure 1: Mesh in 2D.

a) Derive the weak form of equation (1). [2p.]
Solution. Let $V = \{v : \|v\|_{L^2([0,1])} + \|v'\|_{L^2([0,1])} < \infty\}$. Multiply the equation with a test function $v \in V$ integrate over $(0,1)$ and use integration by parts, find $u \in V$ such that,

$$\int_0^1 f(x)v(x)\,dx = \int_0^1 -(a(x)u'(x))'v(x)\,dx \tag{2}$$

$$= \int_0^1 a(x)u'(x)v(x)'\,dx - a(1)u'(1)v(1) + a(0)u'(0)v(0) \tag{3}$$

$$= \int_0^1 a(x)u'(x)v(x)'\,dx + \kappa u(1)v(1) + \kappa u(0)v(0), \tag{4}$$

for all $v \in V$.

b) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a discretization of $(0,1)$. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. Do *not* compute matrix and vector entries. [2p.]
Solution. Let $V_h = \{v \in C_0([0,1]),\ v|_{[x_{i-1},x_i]} \in \mathcal{P}_1([x_{i-1},x_i]),\ i = 1,\ldots,n,\ v(0) = v(1) = 0\} = \text{span}(\{\phi_i\}_{i=0}^n)$. We let $U = \sum_{i=0}^n \xi_i\phi_i$ and test with functions $v = \phi_j$, $j = 0,\ldots,n$ and plug this into the weak form. The finite element method reads, find $\{\xi_i\}_{i=0}^n$ such that,

$$\sum_{i=0}^n \xi_i(a\varphi_i', \varphi_j') + \kappa\xi_n\varphi_j(1) + \kappa\xi_0\varphi_j(0) = (f, \varphi_j), \quad \text{for all } j = 0,\ldots,n.$$

Written on matrix form we have $(A+R)\xi = b$, where $b_j = (f,\phi_j)$, $a_{i,j} = (a\varphi_i', \varphi_j')$, $r_{n,n} = r_{0,0} = \kappa$ and zero otherwise, for all $i,j = 0,\ldots,n$.

c) In the stiffness matrix, terms of the following form appear $\int_0^1 a(x)\varphi_i'(x)\varphi_j'(x)\,dx$. Present two different quadrature rules for approximating $\int_0^1 a(x)\varphi_0'(x)\varphi_0'(x)\,dx$, where $\varphi_0(x)$ is the basis function corresponding to node x_0. Note that a is arbitrary positive. [2p.]
Solution. Let $h_i = x_i - x_{i-1}$, $i = 1,\ldots,n$. Midpoint: $\int_0^1 a(x)\varphi_i'(x)\varphi_j'(x)\,dx \approx \sum_{k=1}^n h_k a(\frac{x_k+x_{k-1}}{2})\varphi_i'(\frac{x_k+x_{k-1}}{2})\varphi_j'(\frac{x_k+x_{k-1}}{2})$.
Trapezoidal rule: $\int_0^1 a(x)\varphi_i'(x)\varphi_j'(x)\,dx \approx \sum_{k=1}^n h_k \frac{a(x_{k-1})\varphi_i'(x_{k-1})\varphi_j'(x_{k-1}) + a(x_k)\varphi_i'(x_k)\varphi_j'(x_k)}{2}$.
Note that the derivative is a constant on each interval so it does not matter where you evaluate it within the interval.

d) Show that,

$$\int_0^1 a(x)|u'(x)|^2\,dx + \kappa|u(1)|^2 + \kappa|u(0)|^2 \le C\int_0^1 |f(x)|^2\,dx,$$

for some constant $C > 0$. You can assume that $\int_0^1 |u(x)|^2\,dx \le C' \int_0^1 |u'(x)|^2\,dx$ for some constant $C' > 0$. [2p.]

Solution. Let $v = u$ in the weak form and apply Cauchy-Schwarz followed by Poincare-Friedrich inequality. We get,

$$\|u\|_a^2 := \int_0^1 a(x)|u'(x)|^2\,dx + \kappa|u(1)|^2 + \kappa|u(0)|^2 \tag{5}$$

$$\le C'^{1/2}\left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}\left(\int_0^1 |u'(x)|^2\,dx\right)^{1/2} \tag{6}$$

$$\le \frac{C'^{1/2}}{a_0^{1/2}}\left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}\left(\int_0^1 a(x)|u'(x)|^2\,dx\right)^{1/2} \tag{7}$$

$$\le \frac{C'^{1/2}}{a_0^{1/2}}\left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}\|u\|_a. \tag{8}$$

The result follows immediatelly.

Problem 3

Consider the problem, find $u(x, y, t)$ such that

$$\begin{aligned} \ddot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \le T \\ u &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ u &= u_0, \quad x \in \Omega, \quad t = 0, \\ \dot{u} &= v_0, \quad x \in \Omega, \quad t = 0, \end{aligned} \tag{9}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) Write equation (9) as a system of two first order (in time) equations. For each fixed $t > 0$, formulate the weak form. [3p.]
Solution. Let $V_0 = \{w : \|w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)} < \infty,\ w|_{\partial\Omega} = 0\}$ and let $v = \dot{u}$. Let $u(0) = u_0$ and $v(0) = v_0$, for each $t > 0$, find $u(t) \in V_0$ and $v(t) \in V_0$ such that, $(\dot{v}, w) + (\nabla u, \nabla w) = 0$, and $(\dot{u}, z) = (v, z)$ for all $w, z \in V_0$. As usual we use Green's formula and that $w = 0$ on $\partial\Omega$.

b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. Let $V_{h,0} \subset V_0 = \{v : \|v\|_{L^2(\Omega)} < \infty, \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$ be the discrete approximation space. [2p.]
Solution. Let $U(0) = \pi u_0$ and $V(0) = \pi v_0$, for each $t > 0$, find $U(t) \in V_{h,0}$ and $V(t) \in V_{h,0}$ such that, $(\dot{V}, w) + (\nabla U, \nabla w) = 0$, and $(\dot{U}, z) = (V, z)$ for all $w, z \in V_{h,0}$. We expand $U(t) = \sum_{i=1}^n \xi_i(t)\varphi_i$ and $V(t) = \sum_{i=1}^n \eta_i(t)\varphi_i$. Let A, with entries $a_{i,j} = (\nabla\varphi_i, \nabla\varphi_j)$, be the stiffness matrix and M, with entries $m_{i,j} = (\varphi_i, \varphi_j)$, be the mass matrix. We plug U and V into the finite element method. We get, $M\dot{\eta} + A\xi = 0$ and $M\dot{\xi} = M\eta$, with given initial conditions.

c) Discretize in time by dividing the time interval $[0, T]$ into N subintervals of equal length. Formulate the Crank-Nicholson method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [3p.]
Solution. Using Crank-Nicholson we get $M\frac{\eta^n - \eta^{n-1}}{k} + A\frac{\xi^n + \xi^{n-1}}{2} = 0$ and $M\frac{\xi^n - \xi^{n-1}}{k} = M\frac{\eta^n + \eta^{n-1}}{2}$. On matrix form we get:

$$\begin{bmatrix} M & \frac{k}{2}A \\ -\frac{k}{2}M & M \end{bmatrix}\begin{bmatrix} \eta^n \\ \xi^n \end{bmatrix} = \begin{bmatrix} M & -\frac{k}{2}A \\ \frac{k}{2}M & M \end{bmatrix}\begin{bmatrix} \eta^{n-1} \\ \xi^{n-1} \end{bmatrix}.$$

Problem 4

Consider the weak form of the Dirichlet problem on a domain $\Omega \subset \mathbf{R}^2$, find $u \in V_0$ such that,

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx, \quad \text{for all } v \in V_0, \tag{10}$$

where $V_0 = \{v : \|v\|_{L^2(\Omega)} < \infty, \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$. Further let $U \in V_{h,0}$ be the finite element approximation of u, where $V_{h,0} \subset V_0$ is the space of continuous piecewise linear functions on a triangulation $\mathcal{K} = \{K\}$ of Ω.

a) Show that,

$$\int_\Omega \nabla(u - U) \cdot \nabla v \, dx = 0, \quad \text{for all } v \in V_{h,0}.$$

[3p.]

b) Show that,

$$\int_\Omega |\nabla(u - U)|^2 \, dx \leq \int_\Omega |\nabla(u - v)|^2 \, dx, \quad \text{for all } v \in V_{h,0}.$$

[3p.]

c) Show that

$$\int_\Omega |\nabla(u - U)|^2 \, dx \leq \sum_{K \in \mathcal{K}} C h_K^2 \|D^2 u\|^2_{L^2(K)},$$

where $h_K = \text{diam}(K)$. The following interpolation estimate can be assumed to be true,

$$\|\nabla(u - \pi u)\|^2_{L^2(K)} \leq C h_K^2 \|D^2 u\|^2_{L^2(K)}.$$

[2p.]

Solutions. See Thm 4.2-4.4 in the December 18, 2009, edition of the lecture notes.

Problem 5

Consider the problem, find $u = u(x)$ such that,

$$-u''(x) + cu(x) = f(x), \quad x \in [0, 1], \tag{11}$$

$$u(0) = u(1) = 0, \tag{12}$$

where $f(x)$ is a given function and $c \geq 0$ is a given constant. Let $0 = x_0 < x_1 < \cdots < x_N = 1$. We denote each subinterval $I_i = [x_{i-1}, x_i]$ and let $h_i = x_i - x_{i-1}$, for all $i = 1, \ldots, N$. We construct a finite element space $V_{h,0} = \{v \in C([0, 1]) : v \text{ linear on } I_i, \text{ for all } i = 1, \ldots, N, v(0) = v(1) = 0\}$.

a) Derive the weak form of equation (11) and the finite element method using the space $V_{h,0}$. [3p.]
Solution. Multiply with a function $v \in V_0$, integrate over $[0, 1]$ and integrate by parts to get, find $u \in V$ such that, $(u', v') + c(u, v) = (f, v)$, for all $v \in V_0$. The finite element method reads: find $U \in V_{h,0}$ such that, $(U', v') + c(U, v) = (f, v)$, for all $v \in V_{h,0}$.

b) Derive the a posteriori error bound for $\left(\|(u - U)'\|^2_{L^2([0,1])} + c\|u - U\|^2_{L^2([0,1])}\right)^{1/2}$. You can assume that $\|(u - U) - \pi_h(u - U)\|_{L^2(I_i)} \leq C h_i \|(u - U)'\|_{L^2(I_i)}$ for all $i = 1, \ldots, N$, where U is the finite element approximation. [3p.]
Solution. Start with the left hand side and use the Galerkin orthogonality we get

by subtracting the finite element formulation from the weak form, and subtract πe in the second slot, and let $e = u - U$,

$$\|e\|_E^2 = \|e'\|^2_{L^2([0,1])} + c\|e\|^2_{L^2([0,1])} = (e', (e - \pi e)') + c(e, e - \pi e) \tag{13}$$

$$= \sum_{i=1}^{n} (-e'' + ce, e - \pi e) \tag{14}$$

$$= \sum_{i=1}^{n} (f + U'' - cU, e - \pi e) \tag{15}$$

$$= \sum_{i=1}^{n} \|f + U'' - cU\|_{L^2(I_i)} \|e - \pi e\|_{L^2(I_i)} \tag{16}$$

$$= C \sum_{i=1}^{n} h_i \|f + U'' - cU\|_{L^2(I_i)} \|e'\|_{L^2(I_i)} \tag{17}$$

$$\leq C \left(\sum_{i=1}^{n} h_i^2 \|f + U'' - cU\|^2_{L^2(I_i)} \right)^{1/2} \|e'\|_{L^2(I)} \tag{18}$$

$$\leq C \left(\sum_{i=1}^{n} h_i^2 \|f + U'' - cU\|^2_{L^2(I_i)} \right)^{1/2} \|e\|_E \tag{19}$$

$$\tag{20}$$

We have integrated by parts on each subinterval and used that $(e - \pi e)(x_i) = 0$ for all $i = 1, \ldots, N - 1$. The statement follows by dividing by $\|e\|_E$ (unless its zero but in that case the result follows immediately).

c) Describe how the a posteriori error bound can be used in an adaptive algorithm to improve the solution U by adding more nodes in the mesh. [2p.]
Solution. See Algorithm 6 on page 40 in the December 18, 2009, edition of the lecture notes. A criteria for marking elements can be found in equation 2.82.

Good luck!
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