UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing
Exam in Finite element methods 2009-08-29

- Time: $9^{00} - 14^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let $I = (0,1)$. Consider the mesh defined by the nodes $x_i = i/3$ for $i = 0, \dots, 3$, with corresponding continuous piecewise linear basis functions $\{\varphi_i\}_{i=0}^3$ defined so that $\varphi_i(x_j) = 1$ if $i = j$ and $\varphi_i(x_j) = 0$ otherwise. Let $V_h = \text{span}(\{\varphi_i\}_{i=0}^3)$ and further let $f(x) = x^3$.

a) Compute the interpolant of f, $\pi_h f \in V_h$. [2p.]

b) Compute the $L^2(I)$ projection of f, $P_h f \in V_h$. [2p.]

c) Show that $P_h f$ is the best approximation of f in V_h measured in the $L^2(I)$ norm. [2p.]

d) Compute a bound for the error $\|f - \pi_h f\|_{L^2(I)} = \left(\int_0^1 |f - \pi_h f|^2 \, dx\right)^{1/2}$. You can use standard interpolation estimates without presenting proofs. [2p.]

Problem 2

Consider the problem, find $u(x)$ such that,

$$
\begin{aligned}
-(a(x)u'(x))' + c(x)u(x) &= f(x), \quad x \in I = (0,1) \qquad (1)\\
u'(0) &= 0,\\
u'(1) &= 0,
\end{aligned}
$$

where $a(x) \geq a_0 > 0$, $c(x) \geq c_0 > 0$, and $f(x)$ are given functions.

a) Derive the weak form of equation (1). [2p.]

b) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a discretization of $(0,1)$. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. Do *not* compute matrix and vector entries. [2p.]

c) Let $a = 1 + x$, $c = 2$, and $x_1 = 0.1$. Compute the first diagonal entry of the stiffness matrix $a_{0,0} = \int_0^1 a(x)\varphi_0'\varphi_0' \, dx + \int_0^1 c(x)\varphi_0\varphi_0 \, dx$, where φ_0 is the finite element basis function corresponding to node x_0. [2p.]

d) Show that, $\|u'\|_{L^2(I)}^2 + \|u\|_{L^2(I)}^2 \leq C\|f\|_{L^2(I)}^2$, for some positive constant C. [2p.]

Problem 3

Consider the problem, find $u(x, y, t)$ such that

$$
\begin{aligned}
\dot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \leq T\\
u &= 0, \quad x \in \partial\Omega, \quad t > 0,\\
u &= u_0, \quad x \in \Omega, \quad t = 0,
\end{aligned}
$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) For each fixed $t > 0$, formulate the weak form. [2p.]

b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. Let $V_{h,0} \subset V_0 = \{v : \|v\|_{L^2(\Omega)} < \infty, \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$ be the discrete approximation space. [3p.]

c) Discretize in time by dividing the time interval $[0, T]$ into N subintervals of equal length. Formulate the Backward Euler method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [3p.]

Problem 4

Consider the weak form of the Dirichlet problem on a domain $I = (0, 1)$, find $u \in V_0$ such that,

$$\int_0^1 u'v'\,dx = \int_0^1 fv\,dx, \quad \text{for all } v \in V_0,$$

where $V_0 = \{v : \|v\|_{L^2(I)} < \infty, \|v'\|_{L^2(I)} < \infty,\ v(0) = v(1) = 0\}$. Further let $U \in V_{h,0}$ be the finite element approximation of u, where $V_{h,0} \subset V_0$ is the space of continuous piecewise linear functions on a mesh $0 = x_0 < \cdots < x_n = 1$.

a) Show that, $\int_0^1 (u - U)'v'\,dx = 0$, for all $v \in V_{h,0}$. [3p.]

b) Show that, $\int_0^1 |(u - U)'|^2\,dx \le \int_0^1 |(u - v)'|^2\,dx$, for all $v \in V_{h,0}$. [3p.]

c) Show that $\int_0^1 |(u - U)'|^2\,dx \le \sum_{K\in\mathcal{K}} Ch_i^2 \|u''\|^2_{L^2(I_i)}$, where $I_i = [x_{i-1}, x_i]$ and $h_i = x_i - x_{i-1}$. The following interpolation estimate can be assumed to be true, $\|(u - \pi u)'\|^2_{L^2(I_i)} \le Ch_i^2 \|u''\|^2_{L^2(I_i)}$. [2p.]

Problem 5

Consider,

$$-\Delta u = f, \quad x \in \Omega,$$
$$\mathbf{n} \cdot \nabla u = g, \quad x \in \partial\Omega,$$

where $\Omega \in \mathbf{R}^2$ has boundary $\partial\Omega$ with unit normal $\mathbf{n}$. The functions f, g are given.

a) Derive the weak form in an appropriate function space. [2p.]

b) Show that $\int_\Omega f\,dx + \int_{\partial\Omega} g\,ds = 0$ is a necessary condition for existence of solution to the weak form. [3p.]

c) Show that if there exists a solution, it can not be unique. [3p.]

Good luck!
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