

- Time: 14⁰⁰ – 19⁰⁰
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Let $I = (0, 1)$. Consider the mesh defined by the nodes $x_i = i/2$ for $i = 0, 1, 2$, with corresponding continuous piecewise linear basis functions $\{\varphi_i\}_{i=0}^2$ defined so that $\varphi_i(x_j) = 1$ if $i = j$ and $\varphi_i(x_j) = 0$ otherwise. Let $V_h = \text{span}(\{\varphi_i\}_{i=0}^2)$ and further let $f(x) = x^3$.

- Compute the interpolant of f , $\pi_h f \in V_h$, as a linear combination of the basis functions. [2p.]
- Compute the $L^2(I)$ projection of f , $P_h f \in V_h$, as a linear combination of the basis functions. [2p.]
- Show that $P_h f$ is the best approximation of f in V_h measured in the $L^2(I)$ norm. [2p.]
- Compute a bound for the error $\|f - \pi_h f\|_{L^2(I)} = (\int_0^1 |f - \pi_h f|^2 dx)^{1/2}$. You can use standard interpolation estimates (with constant $C = 1$) without presenting proofs. [2p.]

Problem 2

Consider the problem, find $u(x)$ such that,

$$\begin{aligned} -(a(x)u'(x))' + c(x)u(x) &= f(x), \quad x \in I = (0, 1) \\ u'(0) &= 0, \\ u'(1) &= 0, \end{aligned} \tag{1}$$

where $a(x) \geq a_0 > 0$, $c(x) \geq c_0 > 0$, and $f(x)$ are given functions.

- Derive the weak form of equation (1). [2p.]
- Let $0 = x_0 < x_1 < \dots < x_n = 1$ be a discretization of $(0, 1)$. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. Do *not* compute matrix and vector entries. [2p.]
- Show that, $\|u'\|_{L^2(I)}^2 + \|u\|_{L^2(I)}^2 \leq C\|f\|_{L^2(I)}^2$, for some positive constant C . [2p.]
- Now, let $a = 1 + x$, $c = 2$, and $x_1 = 0.1$. Compute the first diagonal entry of the stiffness matrix i.e. $a_{0,0} = \int_0^1 a(x)\varphi_0'\varphi_0' dx + \int_0^1 c(x)\varphi_0\varphi_0 dx$, where φ_0 is the finite element basis function corresponding to node x_0 . [2p.]

Problem 3

Consider the problem, find $u(x, y, t)$ such that

$$\begin{aligned} \dot{u} - \Delta u &= 0, \quad x \in \Omega, \quad 0 < t \leq T \\ u &= 0, \quad x \in \partial\Omega, \quad t > 0, \\ u &= u_0, \quad x \in \Omega, \quad t = 0, \end{aligned}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

- a) For each fixed $t > 0$, formulate the weak form. [2p.]
- b) Discretize in space using continuous piecewise linear functions and derive the resulting system of ordinary differential equations. Let $V_{h,0} \subset V_0 = \{v : \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$ be the discrete approximation space. [2p.]
- c) Discretize in time by dividing the time interval $[0, T]$ into N subintervals of equal length. Formulate the Backward Euler method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [2p.]
- d) Show that the $L^2(\Omega)$ norm of the finite element approximation, computed using Backward Euler, $\{U^n\}_{n=0}^N$, decreases in time, i.e., $\|U^n\|_{L^2(\Omega)} \leq \|U^{n-1}\|_{L^2(\Omega)}$ for all $n = 1, \dots, N$. [2p.]

Problem 4

Consider the weak form of the Dirichlet problem on a domain $\Omega \subset \mathbf{R}^2$, find $u \in V_0$ such that,

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \text{for all } v \in V_0,$$

where $V_0 = \{v : \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} < \infty, v|_{\partial\Omega} = 0\}$. Further let $U \in V_{h,0}$ be the finite element approximation of u , where $V_{h,0} \subset V_0$ is the space of continuous piecewise linear functions on a given mesh $\mathcal{T}_h = \cup K$, where K are the triangles of the mesh.

- a) Show that, $\int_{\Omega} \nabla(u - U) \cdot \nabla v \, dx = 0$, for all $v \in V_{h,0}$. [3p.]
- b) Show that, $\|\nabla(u - U)\|_{L^2(\Omega)} \leq \|\nabla(u - v)\|_{L^2(\Omega)}$, for all $v \in V_{h,0}$. [3p.]
- c) Show that, $\|\nabla(u - U)\|_{L^2(\Omega)}^2 \leq \sum_{K \in \mathcal{K}} Ch_K^2 \|D^2 u\|_{L^2(K)}^2$, where $h_K = \text{diam}(K)$. The following interpolation estimate can be assumed to be true, $\|\nabla(u - \pi u)\|_{L^2(K)}^2 \leq Ch_K^2 \|D^2 u\|_{L^2(K)}^2$. [2p.]

Problem 5

Consider the Schrödinger equation,

$$\begin{aligned} i\dot{u} - \epsilon \Delta u &= 0, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \\ u(0) &= u_0, & x \in \Omega. \end{aligned}$$

where $\Omega \in \mathbf{R}^2$ has boundary $\partial\Omega$, $\epsilon > 0$ is a real constant, and i is the imaginary unit.

- a) Let $u = v + iw$ where v, w are real valued functions. Derive the weak form of the resulting system (with unknowns v and w) in appropriate function spaces. [2p.]
- b) Discretize in space using the finite element method on a given triangulation \mathcal{T}_h and in time using Crank-Nicholson on a uniform subdivision of $[0, T]$ into N subintervals of equal length k . Present the resulting linear system of equations. Express matrix entries in terms of the finite element basis functions. [2p.]
- c) Show that the $L^2(\Omega)$ norm of the approximation U^n (using finite elements in space and C-N in time) is independent of time, i.e., $\|U^n\|_{L^2(\Omega)} = \|U^{n-1}\|_{L^2(\Omega)}$, for all $n = 1, \dots, N$. [4p.]

Good luck!
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