UPPSALA UNIVERSITY
Department of Information Technology
Division of Scientific Computing

Exam in Finite element methods 2010-06-10

- Time: $8^{00} - 13^{00}$
- Tools: Pocket calculator, Beta Mathematics Handbook
- Maximum number of points: 40
- All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

Problem 1

Consider the problem, find $u(x)$ such that,

$$\begin{aligned} -(a(x)u'(x))' + c(x)u(x) &= f(x), \quad x \in I = (0,1) \\ u(0) &= 0, \\ a(1)u'(1) &= 0, \end{aligned} \tag{1}$$

where $a(x) \geq a_0 > 0$, $c(x) \geq c_0 > 0$, and $f(x)$ are given functions.

a) Derive the weak form of equation (1). [2p.]

b) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a discretization of I. Derive the finite element method using continuous piecewise linear basis functions. Present the resulting linear system of equations. Do *not* compute matrix and vector entries. [2p.]

c) In the left hand side matrix, terms of the following form appear $\int_0^1 c(x)\varphi_i(x)\varphi_j(x)\,dx$, where $\{\varphi_i\}_{i=1}^n$ is the set of basis functions that spans the finite element space. Present two different quadrature rules for approximating $\int_0^1 c(x)\varphi_0(x)\varphi_0(x)\,dx$, where $\varphi_0(x)$ is the basis function corresponding to node x_0, and $c(x) = 1 + 2x$. [2p.]

d) Show that u is bounded in the $L^2(I)$-norm, $\|u\|_{L^2(I)} = \left(\int_0^1 u(x)^2\,dx\right)^{1/2}$, for arbitrary functions $a(x) \geq a_0 > 0$ and $c \geq c_0 > 0$. [2p.]

Problem 2

Consider the triangulation of a computational domain Ω in Figure 1. Let the nodes be numbered in the following way $N_1 = (0,0)$, $N_2 = (1,0)$, $N_3 = (1,0.5)$, $N_4 = (0.5,0.5)$, $N_5 = (0.5,1)$, and $N_6 = (0,1)$. Let $\{\varphi_i\}_{i=1}^6$ be piecewise linear continuous basis functions fulfilling $\varphi_i(N_j) = 1$ when $i = j$ and $\varphi_i(N_j) = 0$ otherwise. Further let $f(x,y) = sin(\pi x)sin(\pi y)$.

a) Derive the p and t matrix describing the location of the nodes and the triangles of the mesh. Note that you can choose any numbering of the triangles. [2p.]

b) Give an analytical expression for φ_1. [2p.]

c) Compute the nodal interpolant πf as a linear combination of the basis functions $\{\varphi_i\}_{i=1}^6$. [2p.]

d) When computing the L^2-projection Pf the mass matrix with entries $m_{ij} = \int_\Omega \varphi_i\varphi_j\,dx$ needs to be derived. Compute m_{11}. [2p.]

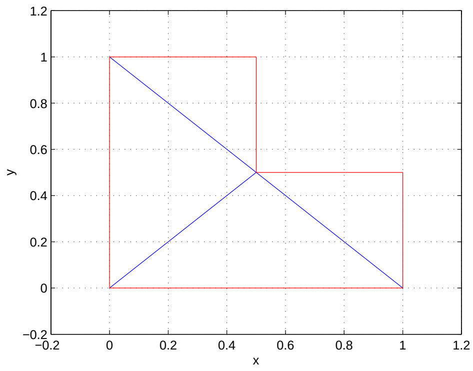

Figure 1: Mesh in 2D.

Problem 3

Let $I = (0, 1)$ and let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a mesh of I. Further let $\{\varphi_i\}_{i=0}^n$ be a set of piecewise linear continuous basis functions so that $\varphi_i(x_i) = 1$ and $\varphi_i(x_j) = 0$ for all $i = 0, \ldots, n$. Let $a_{i,j} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx$ and $m_{i,j} = \int_0^1 \varphi_i(x)\varphi_j(x)\,dx$, and $b_j = \int_0^1 \varphi_j\,dx$, for all $i, j = 1, \ldots, n$. Note that the left end point is not included while the right is. We further let A be the $n \times n$ matrix with entries $a_{i,j}$, M be the $n \times n$ matrix with entries $m_{i,j}$, and b be the $n \times 1$ vector with entries b_j, for $i, j = 1, \ldots, n$. We now formulate the following system of ordinary differential equations,

$$\begin{aligned} M\dot{\chi}(t) + A\chi(t) &= b, \\ \chi(0) &= 0, \end{aligned} \tag{2}$$

where $\chi(t)$ is an unknown $n \times 1$ vector.

a) Equation (2) is a finite element discretization (in space) of a time dependent partial differential equation using continuous piecewise linear basis functions. Formulate the differential equation with boundary conditions and initial condition. [3p.]

b) Discretize equation (2) in time by dividing the time interval $[0, T]$ into N subintervals of equal length. Formulate the Crank-Nicholson method for approximate solution of the system of ordinary differential equations. In particular present the algebraic equation which needs to be solved in each time step. [3p.]

c) Calculate $\lim_{t\to\infty} u_x'(0, t)$, were u is the continuous solution to the partial differential equation derived in a). You can assume that $\int_0^1 u(x, t)\,dx \le C'\|u(t)\|_{L^2(I)} \le C$ for all t, with C independent of t. [2p.]

Problem 4

Consider the weak form of the Dirichlet problem on a domain $\Omega \subset \mathbf{R}^2$, find $u \in V_0$ such that,

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx, \quad \text{for all } v \in V_0, \tag{3}$$

where $V_0 = \{v : \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} < \infty, v = 0 \text{ on } \partial\Omega\}$. Further let $U \in V_{h,0}$ be the finite element approximation of u, where $V_{h,0} \subset V_0$ is the space of continuous piecewise linear functions on a triangulation $\mathcal{K} = \{K\}$ of Ω. Furthermore let $\pi : V_0 \to V_{h,0}$ be an interpolant onto the finite element space.

a) Show that,

$$\|\nabla(u - U)\|_{L^2(\Omega)} \le C \left(\sum_{K \in \mathcal{K}} \rho_K^2 \right)^{1/2}, \tag{4}$$

where,

$$\rho_K^2 = h_K^2 \|f + \Delta U\|^2_{L^2(K)} + \frac{h_K}{4} \|[n \cdot \nabla U]\|^2_{\partial K}.$$

You can assume the trace inequality $\|v\|_{L^2(\partial K)} \le C h_K^{1/2} \|\nabla v\|_{L^2(K)}$, the interpolation estimate $\|v - \pi v\|_{L^2(K)} \le C h_K \|\nabla v\|_{L^2(K)}$, and the Cauchy-Schwarz inequality $(v, w) \le \|v\| \|w\|$ (valid both in the $L^2(\Omega)$ scalar product and for the $\mathbf{R}^n$ scalar product with corresponding norms in the right hand side), to be known. [4p.]

b) Describe how equation (4) can be used in an adaptive algorithm to improve the computed solution using local mesh refinement. [4p.]

Problem 5

Consider the problem, find an eigen-pair $\{u, \lambda\}$ such that,

$$\begin{aligned} -\Delta u + cu &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$.

a) For $c > 0$ show that $\lambda > 0$. [3p.]

b) Introduce an appropriate function space and formulate the weak form of equation (5). [3p.]

c) Let $c \in \mathbf{R}$ be a constant (possibly negative) and $\Omega = [0, 1] \times [0, 1]$. For which values of c is the eigenvalue associated with the eigenfunction $u(x, y) = sin(\pi x) sin(\pi y)$ positive? [2p.]

Good luck!
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