Summation-By-Parts in time
(Project course 15 hp)

Background and focus

In this project we will develop and analyse a numerical method suitable for stiff initial value problems. As a model problem we will consider

$$\begin{array}{ll} Cv_{tt} + Bv_t + Av = G(t) & t > 0 \\ Av = Af_1 \,, v_t = f_2 \,, & t = 0 \end{array} \,, \qquad (1)$$

where A, B and C are $m \times m$ matrices. f_1 and f_2 are initial data, $G(t)$ known data. The model (1) is rather general and could come from a FEM or FD discretisation of an initial-boundary-value-problems (IBVP). A special case is when A is zero, and the problem is reduced (set $w = v_t$) to a first oder ODE system,

$$\begin{array}{ll} Cw_t + Bw = G(t) & t > 0 \\ w_= f_2 \,, & t = 0 \end{array} \,. \qquad (2)$$

In the present study we will restrict the analysis to linear problems. A stable linear model require $C = C^* > 0$, $B + B^* \geq 0$ and $A = A^* \geq 0$. The model (1) is usually rewritten to a first order in time system, and then time-integrated with a Runge-Kutta method.

An A-stable finite difference approximation based on central-difference first derivative summation-by-parts (SBP) operators [2, 12, 4, 6] and the simultaneous approximation term (SAT) method [1] to impose the initial condition for the model (2) is presented in [9, 3]. The SBP-SAT finite difference method is not so easily extended to the more general system (1) on second order form. An attempt was done in [8], but is restricted to central-difference first derivative SBP operators, resulting in a wide-stencil approximation of the second derivative. It can be shown that the SBP-SAT method in [8] is equivalent of solving a first order in time system such as (2).

Project plan

In the present project, we will combine so called upwind SBP operators [5] and the projection method [10, 11, 7] to impose the initial conditions, in order to discretise (1), without reduction to first order in time form. The benefit of upwind SBP operators in contrast to central-difference SBP operators, is their inherent damping of spurious oscillations (see [5] for details). The usage of the projection method also introduce a novel method to impose the initial conditions. Below we set up the project plan:

1. Start by doing a literature study to learn about the SBP-SAT and SBP-Projection methods, starting with [5, 7].

2. Analyse stability for (2) using the energy method. Use the SBP-SAT method to discretise the problem, following the recipe found in [9, 3]. Derive a corresponding discretisation with the SBP-Projection method. Compare both central-difference SBP operators and upwind SBP operators.

3. Analyse stability for (1) using the energy method. Learn about the issues with the SBP-SAT method by reading [8].

4. Extend the SBP-Projection method to (1) and make validation and convergence tests for simple model problems.

5. Do an efficiency comparison against a corresponding RK4 discretisation of (1).

6. Present a numerical study for a more realistic stiff model of the form (1), ideally from a SBP (spatial) discretisation of an IBVP.

The project will be implemented using MATLAB. The SBP operators will be provided.

Relevant courses

The following course is required for the present project: Advanced Numerical Methods.

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