Spectral Analysis for Discretisations of the Diffusion Equation with Variable Coefficients

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Background

A Toeplitz matrix $T_n(f) \in \mathbb{C}^{n\times n}$ is a matrix with constant diagonals:

$$T_n(f) = \begin{bmatrix} \hat{f}_0 & \hat{f}_{-1} & \cdots & \hat{f}_{-(n-1)} \\ \hat{f}_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{f}_{-1} \\ \hat{f}_{n-1} & \cdots & \hat{f}_1 & \hat{f}_0 \end{bmatrix},$$

and for each Toeplitz matrix $T_n(f)$ we can associate a function $f(\theta)$, called a symbol, where the $\hat{f}_k$'s are the Fourier coefficients of f.

The eigenvalues of a Hermitian Toeplitz matrix can be approximated by it's symbol

$$\lambda_j(T_n(f)) = f(\theta_{j,n}) + O(h),$$

which is explained by the theory of Generalised Locally Toeplitz (GLT) sequences.

Matrix-less Method

The Matrix-Less method (MLM) is based on the theory of GLT sequences, and has previously been shown to accurately approximate asymptotic expansions of the spectrum for different types of Toeplitz-like matrices A_n

$$\lambda_j(A_n) = \sum_{k=0}^{\alpha} \tilde{c}_k(\theta_{j,n})\, h^k + O(h^{\alpha+1}), \qquad h = \frac{1}{n+1}.$$

The advantages are that you don't have to construct the full A_n matrix and only need to sample the symbol f or approximated higher order symbols $\tilde{c}_k$ with an evenly spaced grid $\theta_{j,n}$.

What do we want to show?

We would like to numerically investigate whether we can use the above method on matrices which are generated from discretisations of the diffusion equation with variable coefficients defined by a function $a(x)$. That is, if we can use it to approximate the eigenvalues of:

$$A_n = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & \\ & -a_{\frac{5}{2}} & \ddots & \ddots & \\ & & \ddots & \ddots & -a_{n-\frac{1}{2}} \\ & & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix},$$

where $a_j = a(x_j),\ x \in (0,1),\ j = 1, \ldots, n$ and the symbol for this matrix is $a(x)(2 - 2\cos(\theta))$.

Which kind of functions are being considered?

In the paper, a variety of different functions $a(x)$ are tested. Here we show some brief results for a continuous monotone function and a discontinuous function. In the paper we go into more detail about these results and also look to other functions such as functions with a discontinuity in the derivative and complex functions.

References

Sven-Erik Ekström and Carlo Garoni. "A matrix-less and parallel interpolation–extrapolation algorithm for computing the eigenvalues of preconditioned banded symmetric Toeplitz matrices". In: *Numerical Algorithms* 80.3 (Mar. 2018), pp. 819–848. DOI: 10.1007/s11075-018-0508-0.

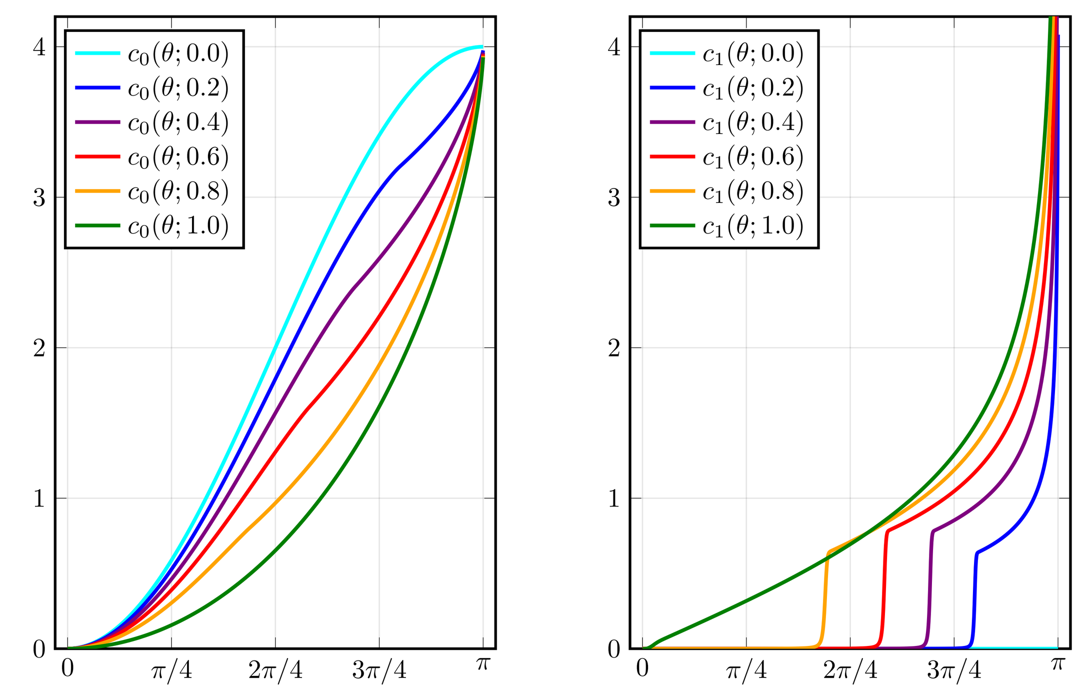
Carlo Garoni and Stefano Serra-Capizzano. *Generalized Locally Toeplitz Sequences: Theory and Applications.* Springer International Publishing, 2018. DOI: 10.1007/978-3-030-02233-4.

Results

Linear Monotone Function

$$a(x) = \epsilon x + (1 - \epsilon)$$

Below we plot the generated $\tilde{c}_0$ and $\tilde{c}_1$ from using the Matrix-less method for a variety of different ϵ.

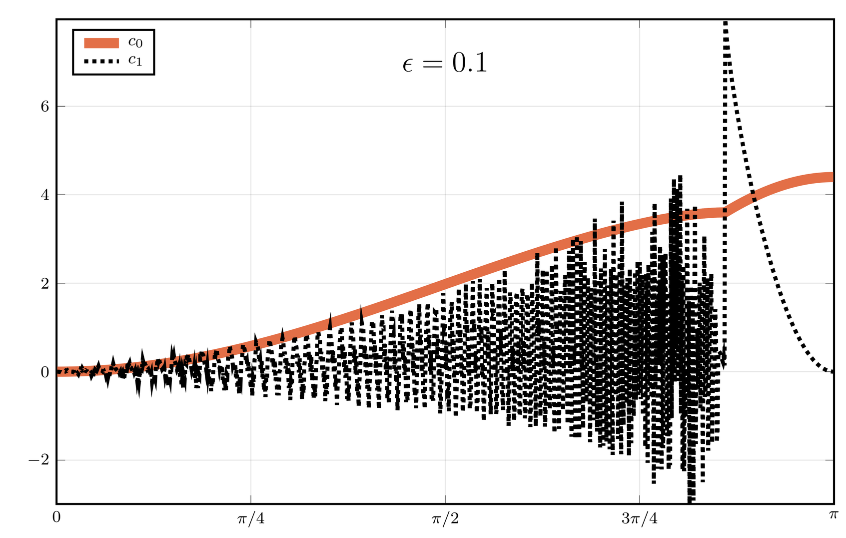


We observe that varying the function from being purely constant to purely linear induces some interesting behaviour in the $\tilde{c}_1$.

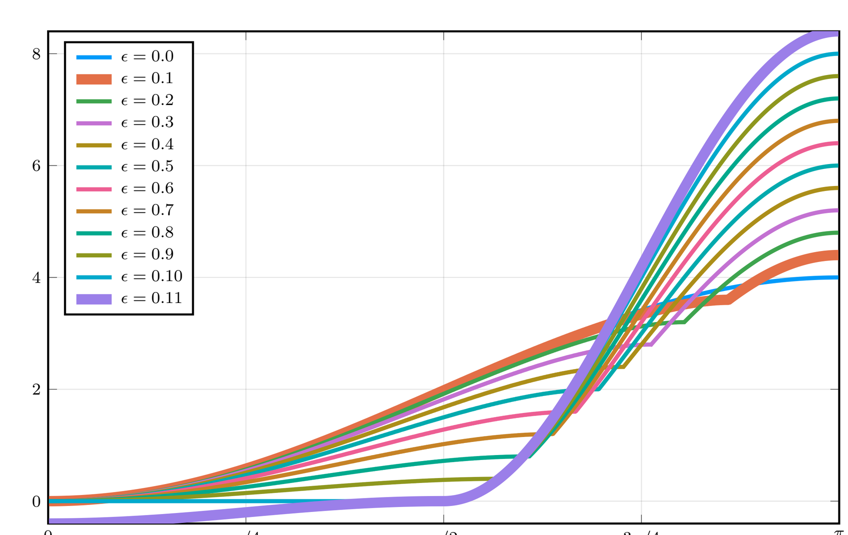
Discontinuous Function

$$a(x) = \begin{cases} 1 - \epsilon, & x \le 0.5 \\ 1 + \epsilon, & x > 0.5 \end{cases}$$

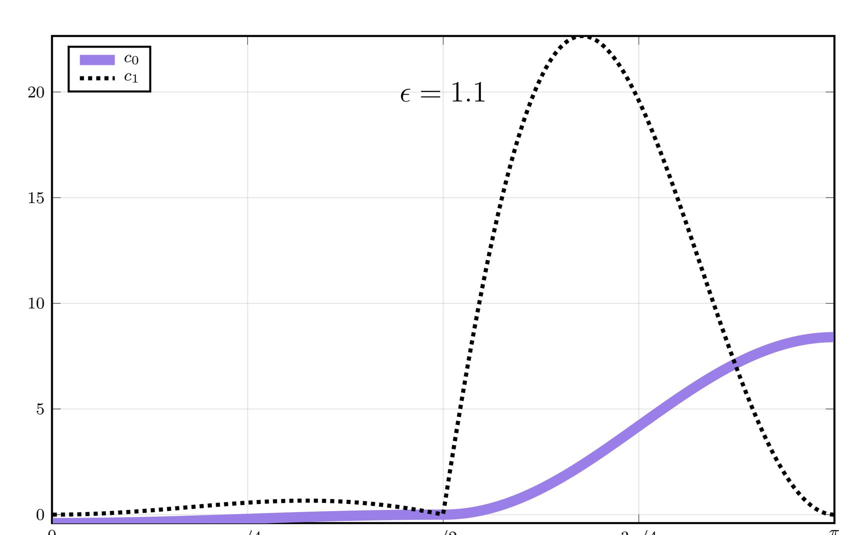
Here we first show the $\tilde{c}_0$ and $\tilde{c}_1$ for $\epsilon = 0.1$.



We observe that there is a 'notch' now in the $\tilde{c}_0$ where the $\tilde{c}_1$ is extremely oscillatory before the notch and smooth after it. If we observe the $\tilde{c}_0$ for varying ϵ shown below:



we can see that the notch moves further along, and thus there are less oscillations along the function if the discontinuity is bigger. We show below the case for $\epsilon = 1.1$ and observe that there are no oscillations in the $\tilde{c}_1$:



Conclusion: The MLM works for matrices derived from discretisations of the diffusion equation with variable coefficients for a wide variety of functions $a(x)$!