Implementing an eigenvalue solver for Toeplitz(-like) matrices

The study of the spectral properties of Toeplitz(-like) matrix sequences is important for the understanding of, for example, discretizations of partial differential equations. Some useful references for this project, regarding the eigenvalues of these matrices, are [1, 2]. Your task in this project will be to implement an eigenvalue (and eigenvector) solver, discussed in [3] by M. K. Ng and W. F. Trench.

An example of a Toeplitz matrix is

$$T_n(f) = \begin{bmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 6 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{1}$$

that is all diagonals are constants. The matrix (1) is the bi-Laplace matrix from the second order finite difference discretization of the fourth derivative. An example of Toeplitz-like matrices is a Toeplitz matrix plus a low-rank matrix. Also, the elements of the matrix can be matrices of fixed or varying size, as the matrix size increases.

This project is focused on implementation and programming. The motivation of this study is mainly to (a) understand the method and its efficiency compared to other methods and (b) possibly integrate current research knowledge about Toeplitz-like matrices to further improve the method, or to improve other methods, like matrix-less methods [1].

Planned Tasks (many extensions are possible)

1. Understand the basics of the theory of generalized locally Toeplitz (GLT) sequences [1, 2].
2. Implement the algorithm described in [3] in Julia [4] (or possibly Matlab).
3. If time permits, parallelize the algorithm.
4. Compare the efficiency of the implemented method with banded eigenvalue solvers in LAPACK/MKL and also for matrix-less solvers [1]. Relevant model matrices will be provided.

Practical details

- Advisor: Sven-Erik Ekström, Assistant professor at the division of scientific computing, IT department, Uppsala University, `sven-erik.ekstrom@it.uu.se` (`www.2pi.se`).
- Prerequisites: Basic understanding of linear algebra. Good programming skills.
- Language is Julia [4] (or possibly Matlab). Previous experience is a plus, but not required.
- Meetings and discussions to be carried out on platforms like IRC, Slack, Discord, or Zoom.

References

[1] S.-E. Ekström, *Matrix-Less Methods for Computing Eigenvalues of Large Structured Matrices*, Ph.D. Thesis, Uppsala University (2018) (`www.2pi.se/thesis.pdf`)
[2] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Springer, 2017 (`www.doi.org/10.1007/978-3-319-53679-8`)
[4] M. K. Ng and W. F. Trench, *Numerical solution of the eigenvalue problem for Hermitian Toeplitz-like matrices* Technical report TR-CS-97-14, Australian National University (1997) (link)
[5] J. Bezanson, A. Edelman, S. Karpinski, and V. Shah, *Julia: A fresh approach to numerical computing*, SIAM review 59:1, pp. 65–98 (2017) (`www.julialang.org`)