Hidden structures in the eigenvectors of certain matrix sequences and experience in Julia

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The question how to compute eigenvalues and eigenvectors of matrices has been and remains one of the most studied for many decades. There is good existing software, however it remains a difficult and computationally heavy task. One particular question we can pose is how to compute the eigenvector to a known (exact) eigenvalue in a cheap and reliable way. This project addresses the latter issue and even more, it aims at detecting structures in the eigenvalues for a particular class of matrices, Toeplitz matrices. Generally speaking, series of Toeplitz matrices and their spectrum can be characterized by an analytical function, referred to as the *symbol* or the *generating symbol* of the matrix sequence.

$$\underset{\text{Toeplitz matrix}}{T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ -1 & -1 & 1 & 2 \\ -3 & -2 & -1 & 1 \end{bmatrix}} \qquad \underset{\text{Toeplitz-like matrix}}{\widetilde{T} = \begin{bmatrix} 5 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ -1 & -1 & 1 & 2 \\ -3 & -2 & -1 & 1 \end{bmatrix}} \tag{1}$$

Toeplitz matrices arise in a broad spectrum of application problems. The study of the spectral properties of Toeplitz(-like) matrix sequences is important for the understanding of, for example, discretizations of partial differential equations. Some useful references for this project, regarding the *eigenvalues* of these matrices, are [1, 2]. Your task in this project will be to study the *eigenvectors* of some particular Toeplitz(-like) matrix sequences.

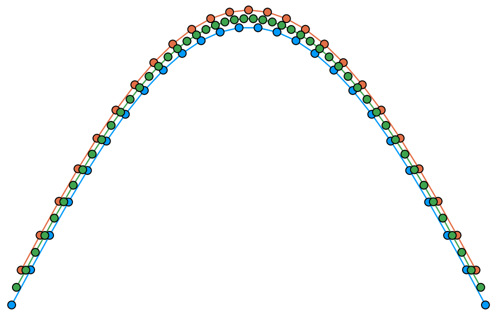
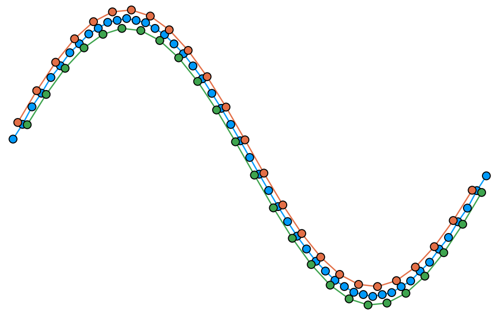
An example of a Toeplitz matrix that you will study in this project is

$$T_n(f) = \begin{bmatrix} 6 & -4 & 2 & & & & \\ -4 & 6 & -4 & 2 & & & \\ 2 & -4 & 6 & -4 & 2 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 2 & -4 & 6 & -4 & 2 \\ & & & 2 & -4 & 6 & -4 \\ & & & & 2 & -4 & 6 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2}$$

and the associated matrix sequence (of matrices with increasing size $n \times n$) is denoted $\{T_n(f)\}_n$. Here f is the so called generating symbol, and is equal to $f(\theta) = 6 - 8\cos(\theta) + 2\cos(2\theta)$. The symbol describes the eigenvalue distribution [1, 2].

Similar structures exists also for the eigenvectors, but are less studied, but two relevant reports for this project are [4, 5]. You will further explore the (hidden) eigenstructures present in these sequences. For example, you can explore one or more of the following main questions,

- Can we implement Ng-Trench [3] (see also [4])for general Toeplitz-like matrices, and for block-Toeplitz matrices?
- When using matrix-less methods to approximate the eigenvectors [4] the eigenvectors with high indices will be highly oscillating and we can not approximate them well. Can we "rotate" the spectrum to smooth out oscillations for any given eigenvalue?
- Modifying the symbol f above to the symbol $f(\theta) = 6 - 8\cos(\theta) + 2\gamma\cos(2\theta)$, where $\gamma \notin [-1, 1]$ will yield a *non-monotone symbol*, where the ordering of the eigenvalues is not unique. Can we use the eigenvectors to correctly order eigenvalues for matrices generated by non-monotone symbols?
- Can we find hidden interlaced eigenvector symbols, as indicated in the figures below that arise from the matrix given above.

This project relies heavily on experimental mathematics to gain insights on pure linear algebra topics. The focus can be theoretical, implementation focused, or fully exploratory, and shall be decided by the students and advisors in the beginning of the project.

Planned Tasks (many extensions are possible)

1. Understand the basics of the theory of generalized locally Toeplitz (GLT) sequences [1, 2].
2. Pick a main focus of the project, understand the question at hand and develop numerical tools to further our understanding. Formulate hypothesis and conjectures, and proper experiments to verify them.
3. Write a report of your findings.

Practical details

- Prerequisites: Basic understanding of linear algebra. Creativity, patience, and independence to propose, implement, and execute numerical experiments and present relevant data.
- Programming language is JULIA [6], no previous experience required.
- Meetings and discussions are mainly carried out on Discord and Zoom.

References

[1] S.-E. Ekström, *Matrix-Less Methods for Computing Eigenvalues of Large Structured Matrices*, Ph.D. Thesis, Uppsala University, 2018 (`www.2pi.se/thesis.pdf`)
[2] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Springer, 2017 (`www.doi.org/10.1007/978-3-319-53679-8`)
[3] M. K. Ng and W. F. Trench, Numerical solution of the eigenvalue problem for Hermitian Toeplitz-like matrices, 1997.
`https://openresearch-repository.anu.edu.au/bitstream/1885/40750/3/TR-CS-97-14.pdf`
[4] D. Meadon, *A Matrix-less Method for Approximating the Eigenvectors of Toeplitz-like Matrices*, M.Sc. Thesis, Uppsala University, 2021 (`http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-458841`)
[5] F. Cers, O. Groth, and M. Knebel, *Exploring and extending eigensolvers for Toeplitz(-like) matrices*, Uppsala University, 2022 (`http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-477693`)
[6] J. Bezanson, A. Edelman, S. Karpinski, and V. Shah, *Julia: A fresh approach to numerical computing*, SIAM review 59:1, pp. 65–98 (2017) (`www.julialang.org`)