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Pricing European Options using stochastic volatility with integral jump term

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PROJECT REPORT

Abstract

Traditional methods for pricing options (like Black-Scholes), do not take sudden changes in underlying assets into account. Even models with stochastic volatility does not produce valid enough results. The solution is Merton's jump model. Combining Merton's jump model with Heston's stochastic volatility model gives Bates model for pricing options. Bates model is in this project studied in comparison with Heston's model.

Bates model is more accurate than Heston's stochastic volatility model but is computationally more demanding. With jump term the computational requirements grows exponentially as mesh size increase, while not using jump term the computational power increases linearly while the increasing number of degrees of freedom and mesh size increase. In conclusion Bates model in comparison with Heston's model is a trade-off in between accuracy and efficiency.

Contents

1 Introduction

An option is a contract to buy or sell a specific financial product, referred to as the option's underlying instrument or underlying asset. The contract itself is very precise. It establishes a specific price, called the strike price, at which the contract may be exercised, or acted on. It has an expiration date, at which it no longer has any value and no longer exists. There are two main types of options; call-options and put-options. You can buy or sell either type. You make a choice; whether to buy or sell and whether to choose a call or a put, based on what you want to achieve as an options investor.

For the two main types (put or call), there are two variants; American and European. When buying a call, you have the right to buy the underlying instrument at the strike price on or before the expiration date for American options, or on the expiration date for European options. If you buy a put, you have the right to sell the underlying instrument on or before expiration for American options, or on the expiration date for European options. In either case, as the option holder, you also have the right to sell the option to another buyer during its term or to let it expire worthless. What a particular options contract is worth to a buyer or seller is measured by how likely it is to meet their expectations.

For example a call option is worth something if the current market value of the underlying asset is above the exercise price of the option, and worth nothing or less if the stock is below the exercise price at expiration and/or before for American options.

To determine the price of an option one can use Black-Scholes model to evaluate the price. Black-Scholes model is shown below:

$$dx_t = \mu x_t \, dt + \sigma x_t \, dW_t \tag{1}$$

where x_t is the price of the underlying asset, σ is the volatility, μ is the drift of x and W_t is a brownian motion. This model can in some cases be solved using analythical methods when refering to european options. However if we want to reproduce more realistic paths for the price of the underlying assets and option price we have to use more advanced methods, such as Heston's model which takes stochastic volatility into account.

Heston's model:

$$\begin{aligned} dx_t &= \mu x_t \, dt + \sqrt{y_t} x_t \, dW_t^1 \\ dy_t &= \alpha(\beta - y_t) \, dt + \gamma \sqrt{y_t} \, dW_t^2 \end{aligned} \tag{2}$$

Here y_t is the volatility, α is the rate at which the variance reverts back to the value β and γ is the volatlility of variance. The original Black-Scholes model of the market for an equity makes some explicit assumptions:

- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- The price follows a geometric Brownian motion with constant drift and volatility.
- There are no transaction costs.
- The stock does not pay a dividend.
- All securities are perfectly divisible. (i.e. it is possible to buy any fraction of a share).
- There are no restrictions on short selling [2].

The model only treats European-type options. From these ideal conditions in the market for an equity (and for an option on the equity), Fischer Black and Myron Scholes [9] show that the value of an option (the Black-Scholes formula) varies only with the stock price and the time to expiry. Thus it is possible to create a hedged position [1], consisting of a long position in the stock and a short position in (calls on the same stock), which value will not depend on the price of the stock. Below Black-Scholes model is shown in PDE form:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} = ru \tag{3}$$

where u is the price of the option on the equity, x is the price of the equity, t is the time, σ is the volatility of the equity and r the "risk-free interest rate".

Numerical simulations and analysis have shown that traditional pricing of options using Black-Scholes and constant volatility does not produce valid enough results. Increasing accuracy with Heston's model for stochastic volatility might solve the problem, but still one aspect of the asset value isn't taken into account. Namely jumps in the price of the underlying asset. A jump is a large and sudden change in the price. Merton's model [7] takes jump into account.

When jumps occur increasing the overall volatility might solve the problem since the higher risk is taken into account. But what if the volatility of the underlying asset is relatively low and a jump is an individual occurrence?

Bates model [3] with stochastic volatility and integral jump term treats volatility and jumps separately thus hopefully produce a accurate enough result when for instance the overall volatility is low but jumps occur. We will take a closer look at bates model in the next section.

2 Pricing Model

Bate's model combines the Heston stochastic volatility model [4] with the Merton jump model [8]. The differential equation describing Bate's model is

$$\begin{aligned} dx_t &= \mu x_t\,dt + \sqrt{y_t}x_t\,dW_t^1 + (J-1)x_t\,dn \\ dy_t &= \alpha(\beta - y_t)\,dt + \gamma\sqrt{y_t}\,dW_t^2 \end{aligned} \tag{4}$$

It looks almost the same as (2) with the added term on the first row, where the Poisson arrival process n has the rate λ. The jump rate J is taken from a distribution

$$f(J) = \frac{1}{\sqrt{2\pi}\delta J}\exp\left(-\frac{[\ln J - (\gamma - \delta^2/2)]^2}{2\delta^2}\right) \tag{5}$$

where γ is the mean and δ the variance of the jump.

This gives us the following in PDE form (same as in Toivanen [10])

$$\begin{aligned} \frac{\partial u}{\partial \tau} = (r - q - \lambda\alpha)\frac{\partial u}{\partial x} + \frac{1}{2}yx^2\frac{\partial^2 u}{\partial x^2} + \rho\gamma xy\frac{\partial^2 u}{\partial x \partial y} + (\alpha(\beta - y))\frac{\partial u}{\partial y} \\ -(r+\lambda)u + \frac{1}{2}\gamma\frac{\partial^2 u}{\partial y^2} + \lambda\int_0^\infty u(Jx, y, \tau)f(J)dJ := Lu \end{aligned} \tag{6}$$

where $\tau = T - t$ is the time to expiry and q the dividend yield. We need to truncate the unbounded domain for numerical reasons.

$$x \in (0, X), \quad y \in (0, Y), \quad \tau \in (0, T). \tag{7}$$

The value of u at the expiry time T is given by

$$u(x, y, T) = g(x, y) \tag{8}$$

where the payoff function g depends on the type of option. Here we use call options so the payoff function with strike price K is

$$g(x, y) = max\{x - K, 0\}. \tag{9}$$

But due to our change of variables to τ the payoff function g actually gives us the initial value for u.

Boundary conditions used for call options.

$$\begin{aligned}
u(0, y, \tau) &= g(0, y) \\
\frac{\partial u(x, Y, \tau)}{\partial y} &= 0 \\
u(X, y, \tau) &= g(X, y) \\
\frac{\partial u(x, 0, \tau)}{\partial y} &= 0
\end{aligned} \tag{10}$$

3 Discretization

We have used the same space discretization as [6] for the derivatives and for the integral term we use the trapezoidal-rule. For further reference $u_{i,j} = u(x_i, y_j),\ 1 \le i \le n,\ 1 \le j \le m$ and $h_x = \dfrac{X}{n}$, $h_y = \dfrac{Y}{m}$ are the grid steps in the x- and y-directions.

3.1 Discretization of the PDE operator

We discretize the PDE part of L in (6) with the help of finite differences.

$$L_{PDE}u = au_{xx} + bu_{yy} + cu_{xy} + du_x + eu_y + fu \tag{11}$$

where

$$\begin{gathered}
a = \frac{1}{2}yx^2, \quad b = \frac{1}{2}\gamma, \quad c = \rho\gamma xy, \\
d = (r - q - \lambda\alpha), \quad e = \alpha(\beta - y), \quad f = -(r + \lambda)
\end{gathered} \tag{12}$$

For completenes we recall the standard central difference approximations for the first and second derivatives of u, which occur in 11:

$$
\begin{aligned}
u_x &\approx \frac{u_{i+1,j} - u_{i-1,j}}{2h_x} \\
u_y &\approx \frac{u_{i,j+1} - u_{i,j-1}}{2h_y} \\
u_{xx} &\approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} \\
u_{yy} &\approx \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} \\
u_{xy} &\approx \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h_y h_x}
\end{aligned}
\tag{13}
$$

Using (13) to discretize (11) we obtain a 9-point discretization stencil for each $u_{i,j}$, the coefficients are shown in Figure 1. After the discretization of (11) we obtain a linear system of equations of the form

$$A\underline{u} = b \tag{14}$$

where A is a nonsymmetric sparse matrix with a 9-diagonal structure.

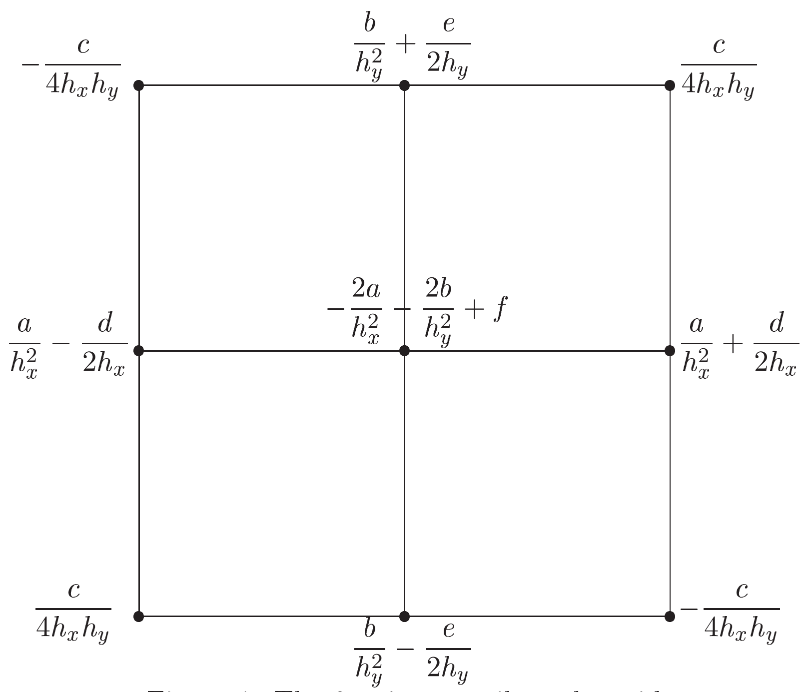

Figure 1: The 9 point stencil on the grid

3.2 Discretization of the integral term

The integral term in (6) needs to be evaluated at each grid point, $x = x_i,\ 1 \leq i \leq n$, i.e we have to compute

$$I_i = \int_0^\infty u(Jx_i, y, \tau) f(J) dJ \tag{15}$$

A change of variables to $J = e^s$ and $dJ = e^s ds$ transforms I_i into a new form,

$$I_i = \int_{-\infty}^\infty u(e^s x_i, y, \tau) p(s) ds \tag{16}$$

where

$$p(s) = \frac{1}{\sqrt{2\pi}\delta} \exp\left(-\frac{[s - (\gamma - \delta^2/2)]^2}{2\delta^2}\right) \tag{17}$$

is the transformed $f(J)$ function.

We now decompose the integral as a sum of integrals over the grid intervals

$$\begin{aligned} I_i = & \int_{ln\frac{x_1}{x_i}}^{ln\frac{x_2}{x_i}} u(e^s x_i, y, \tau) p(s) ds + \int_{ln\frac{x_2}{x_i}}^{ln\frac{x_3}{x_i}} u(e^s x_i, y, \tau) p(s) ds + ... \\ & + \int_{ln\frac{x_{n-1}}{x_i}}^{ln\frac{x_n}{x_i}} u(e^s x_i, y, \tau) p(s) ds + \int_{ln\frac{x_n}{x_i}}^{\infty} g(e^s x_i, y) p(s) ds \end{aligned} \tag{18}$$

The latter can be written as

$$I_i = \sum_{k=1}^{n-1} I_{i,k} + \int_{ln\frac{x_n}{x_i}}^{\infty} \overbrace{g(e^s x_i, y) p(s) ds}^{\approx 0} \tag{19}$$

where k is referring to the intervals we integrate over.

$$I_{i,k} = \int_{ln\frac{x_k}{x_i}}^{ln\frac{x_k+1}{x_i}} u(e^s x_i, y, \tau) p(s) ds \tag{20}$$

We evaluate each $I_{i,k}$ using the trapezoidal rule.

$$
\begin{aligned}
I_{i,k} \approx \tfrac{1}{2}\Big[u(\underbrace{e^{ln\frac{x_k}{x_i}}x_i}_{=x_k}, y, \tau)p(ln\tfrac{x_k}{x_i})+ \\
+u(\underbrace{e^{ln\frac{x_{k+1}}{x_i}}x_i}_{=x_{k+1}}, y, \tau)p(ln\tfrac{x_{k+1}}{x_i})\Big](ln\ x_{k+1} - ln\ x_i - (ln\ x_k - ln\ x_i))
\end{aligned} \tag{21}
$$

Next we explain how to evaluate the term $I_{i,1}$ in (20), which needs a separate consideration since $ln\dfrac{x_1}{x_i} = ln\dfrac{0}{x_i} = -\infty$. We obtain the integral

$$
\int_{-\infty}^{ln\frac{x_2}{x_i}} u(e^s x_i, y, \tau)p(s)ds \approx u(x_i, y, \tau)\int_{-\infty}^{ln\frac{x_2}{x_i}} p(s)ds \tag{22}
$$

where

$$
\begin{aligned}
\int_{-\infty}^{ln\frac{x_2}{x_i}} p(s)ds &= \int_{-\infty}^{ln\frac{x_2}{x_i}} \frac{1}{\sqrt{2\pi}\delta}\, e^{\frac{-[s-(\gamma-\delta^2/2)]^2}{2\delta^2}} ds \\
= \left\{ \begin{array}{c} t = \frac{s-(\gamma-\delta^2/2)}{\sqrt{2}\delta} \\ dt\sqrt{2}\delta = ds \end{array} \right\} &= \frac{1}{\sqrt{2\pi}\delta} \int_{-\infty}^{\frac{ln\frac{x_2}{x_i}-(\gamma-\delta^2/2)}{\sqrt{2}\delta}} \sqrt{2}\delta e^{-t^2} dt \\
= \left\{ \alpha = \frac{ln\frac{x_2}{x_i}-(\gamma-\delta^2/2)}{\sqrt{2}\delta} \right\} &= \frac{1}{\sqrt{\pi}} \int_{-\alpha}^{\infty} e^{-t^2} ds \\
= \frac{1}{2}\ erfc(\alpha) &= \frac{1}{2}(1 - erf(\alpha))
\end{aligned} \tag{23}
$$

The special case when $i = 1$, i.e. $x_1 = 0$, is resolved as follows:

$$
\begin{aligned}
I_0 = \int_0^{\infty} u(Jx_1, y, \tau)f(J)dJ \\
= u(0, y, \tau)\underbrace{\int_0^{\infty} f(J)dJ}_{=1} = u(0, y, \tau)
\end{aligned} \tag{24}
$$

The discretization of the integral will give us essentially full blocks on the diagonal of the matrix corresponding to the integral term. Figure 2 illustrates the structure and the values of the entries in one of these blocks.

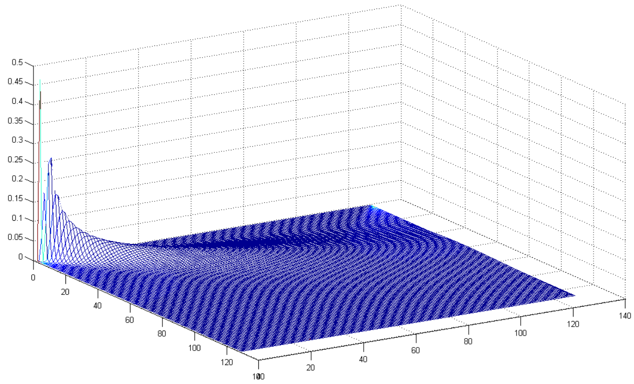

Figure 2: Mesh-plot of a block I_i

3.3 Time discretization

We use the BDF-2 time stepping-scheme as in [5] to solve the system. We use Euler-backward for the first time-step since BDF-2 is a multi-step method.

Euler-backward

$$u_{i,j}^{n} = Au_{i,j}^{n} + u_{i,j}^{n-1} \tag{25}$$

where A is the space discretization of the operator L. Thus A is a block-diagonal matrix with essentially full blocks due to the integral term.

BDF-2

$$\frac{3}{2}u_{i,j}^{n} = kAu_{i,j}^{n} + 2u_{i,j}^{n-1} - \frac{1}{2}u_{i,j}^{n-2} \tag{26}$$

where k is the size of the time-step. This gives us a large system of equations to solve for each time-step, especially since A has blocks on the diagonal.

4 Tasks

Bates model [3] is implemented in matlab 7 using BDF-2 and Euler-backward. All results in the following sections are based on this matlab implementation.

5 Results

All results are based on the following values:

- $x_{min} = 0$

- $x_{max} = 400$
- $r = 0.03$ (The risk-free interest)
- $K = 100$ (the strike price)
- $T = 0.5$ (time domain)
- $q = 0.05$ (dividend yield)
- $\alpha = 2.0$ (the rate of reversion of the mean level)
- $\beta = 0.04$ (the mean level of variance)
- $\gamma = 0.25$ (the volatility of the variance)
- $\rho = -0.5$ (the correlation between the price and variance process)
- $y_{min} = 0$
- $y_{max} = 1$
- $\gamma_{integral} = -0.5$ (the mean jump)
- $\delta_{integral} = 0.4$ (the variance of jump)
- $\lambda_{integral} = 0.5$ (the jump rate)

When using only stochastic volatility computations are quite fast and relatively accurate. That is for one underlying asset.

The model can easily be expanded to account for several underlying assets which will dramatically change the computational time.

When adding the jump term to the matrix the computational time increases exponentially with increasing mesh size. This is shown in figure 3.

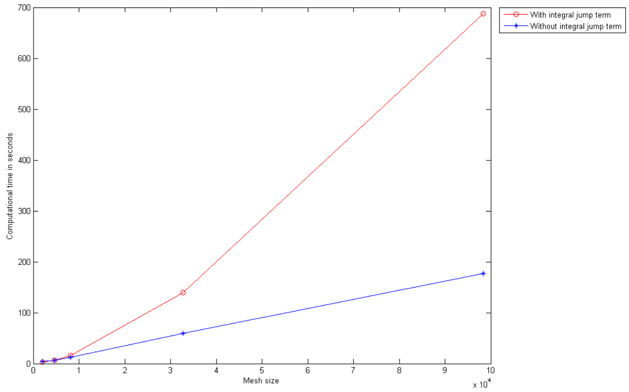

Figure 3: Computational-time for stochastic volatility only in comparison with stochastic volatility and jump term.

When the volatility of the underlying asset is low the difference in option price is significant. That is when comparing with and without jump term (as shown in Table 1 and Table 2). When the volatility of the underlying asset is high the difference in option price is very small. That is when comparing with and without jump term (as shown in Table 1 and Table 2).

Asset price	80	90	100	110	120
Volatility=0.031746	0.023639	0.089502	0.27108	0.68951	1.5288
Volatility=0.49206	0.86764	1.7811	3.211	5.2471	7.9524
Volatility=0.74603	1.95167	3.44939	5.52624	8.21954	11.547

Table 1: Option price without integral jump. Mesh size n=128, m=64.

Asset price	80	90	100	110	120
Volatility=0.031746	0.039448	0.12524	0.34215	0.81752	1.7422
Volatility=0.49206	0.95432	1.9393	3.4741	5.6543	8.5484
Volatility=0.74603	2.11189	3.72011	5.94851	8.83842	12.4105

Table 2: Option price with integral jump. Mesh size n=128, m=64.

The result of calculating the price for a European call option using stochastic volatility and integral jump term is shown in figure 4.

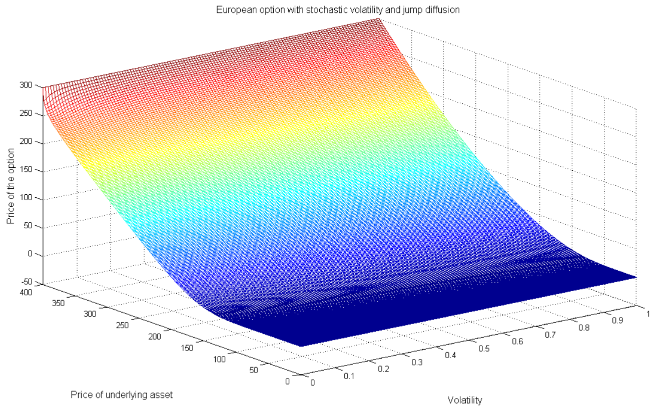

Figure 4: Plot showing call option with integral jump term. Mesh size n=256, m=128.

6 Conclusion

The traditional Black-Scholes model gives a straight forward and quick result on how to price an option. However the result may not be accurate enough in a highly competitive market.

Heston's model with stochastic volatility (which is widely spread), solves many of the accuracy problems associated with the traditional Black-Scholes model. However sudden changes in underlying asset value are not taken into account. (This could for example be the sudden fall of a stock.)

Bates model with integral jump term solves the problem of taking sudden changes (jumps) in the underlying asset into account, and gives a more accurate option price.

While Bates model produces a very accurate result the increase in computational time is significant.

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