Systems and Control
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Introduction to computer control systems: Selected exercises for the problem solving sessions

Master program in embedded systems, period 2, 2010

Assignment: Solve the exercises listed below individually for next Wednesday (2010/12/15).

Problem solving session X (Ex10)

1. Given the Volterra predator-prey mathematical model

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_2 - x_1 x_2 \end{aligned}$$

   (a) Obtain the linear space-state representation at the equilibrium points: $(x_{10}, x_{20}) = (0, 0)$ and $(x_{10}, x_{20}) = (1, 1)$.

   (b) Are the equilibrium points stable?

2. Consider the continuous-time linear system given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (0 \qquad 1)x \end{aligned}$$

   with a transition matrix given by

$$\Phi = \begin{pmatrix} e^{-t}(1-t) & -te^{-t} \\ te^{-t} & e^{-t}(1+t) \end{pmatrix}$$

   (a) Obtain the output response of the system, $y(t)$ (for $t > 0$), if a unit step input is applied at $t_0 = 0$ and the initial conditions are $x(0) = [1, 0]^T$.

   (b) Obtain the transfer function of the system.

   (c) Design a PID controller so that the closed-loop poles are at $(p_1, p_2, p_3) = (-4 \pm 5j, -6)$. Obtain the values k_P, k_I and k_D.

$$H(p) = k_P + \frac{k_I}{p} + k_D p$$

3. Consider a continuous-time linear system given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (2 \quad 5)x \end{aligned}$$

(a) Sample the system with a sampling time T=0.1.

(b) Is the sampled system observable and controllable?

(c) Design an observer for the discrete-time system so that the poles of the observer are in $\lambda_{1,2}$=0.35.

(d) Obtain a control law $u(t) = -K\hat{x}(t) + Hr(t)$ ($\hat{x}(t)$ are the estimated states and $r(t)$ is the reference signal), so that the poles of the controller are in $\lambda_{1,2}$=0.5 and the static gain of the closed-loop system is 1.

Solutions for the problem solving session X (Ex10)

1. Given the Volterra predator-prey mathematical model

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_2 - x_1 x_2 \end{aligned}$$

   (a) Obtain the linear space-state representation at the equilibrium points:

      i. $(x_{10}, x_{20}) = (0, 0)$

$$\begin{pmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$$

      with $\bar{X}_1 = x_1 - x_{10}$ and $\bar{X}_2 = x_2 - x_{20}$

      ii. $(x_{10}, x_{20}) = (1, 1)$

$$\begin{pmatrix} \dot{\bar{X}}_1 \\ \dot{\bar{X}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$$

      with $\bar{X}_1 = x_1 - x_{10}$ and $\bar{X}_2 = x_2 - x_{20}$

   (b) Are the equilibrium points stable?

      i. Eigenvalues are $\lambda_{1,2} = \pm 1$. The equilibria point is unstable.

      ii. Eigenvalues are $\lambda_{1,2} = \pm j$. The equilibria point is marginally stable.

2. Consider the continuous-time linear system given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y &= (0 \quad 1) x \end{aligned}$$

   with a transition matrix given by

$$\Phi = \begin{pmatrix} e^{-t}(1-t) & -te^{-t} \\ te^{-t} & e^{-t}(1+t) \end{pmatrix}$$

   (a) Obtain the output response of the system, $y(t)$ (for $t > 0$), if a unit step input is applied at $t_0 = 0$ and the initial conditions are $x(0) = [1, 0]^T$.

$$y(t) = 1 - e^{-t}, \qquad t > 0$$

(b) Obtain the transfer function of the system.

$$G(s) = \frac{1}{(s+1)^2}$$

(c) Design a PID controller so that the closed-loop poles are at $(p_1, p_2, p_3) = (-4 \pm 5j, -6)$. Obtain the values k_P, k_I and k_D.

$$H(p) = k_P + \frac{k_I}{p} + k_D p$$

$$\begin{aligned} k_P &= 88 \\ k_I &= 246 \\ k_D &= 12 \end{aligned}$$

3. Consider a continuous-time linear system given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u \\ y &= (2 \quad 5)x \end{aligned}$$

(a) Sample the system with a sampling time T=0.1.

$$\begin{aligned} x(t+T) &= Fx(t) + Gu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$\begin{aligned} x(t+T) &= \begin{pmatrix} 0.6703 & 0 \\ 0 & 0.8187 \end{pmatrix} x(t) + \begin{pmatrix} 0.0824 \\ 0.0906 \end{pmatrix} u(t) \\ y(t) &= (2 \quad 5)x(t) \end{aligned}$$

(b) Is the sampled system observable and controllable?
Controllability matrix:

$$\mathcal{S} = \begin{pmatrix} 0.0824 & 0.0552 \\ 0.0906 & 0.0742 \end{pmatrix}$$

The system is controllable
Observability matrix:

$$\mathcal{O} = \begin{pmatrix} 2 & 5 \\ 1.3406 & 4.0935 \end{pmatrix}$$

The system is observable

(c) Design an observer for the discrete-time system so that the poles of the observer are in $\lambda_{1,2}$=0.35.

Observer:

$$\hat{x}(t+T) \;=\; A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t))$$

Observer gain:

$$L \;=\; \begin{pmatrix} -0.3457 \\ 0.2961 \end{pmatrix}$$

(d) Obtain a control law $u(t) = -K\hat{x}(t) + Hr(t)$ ($\hat{x}(t)$ are the estimated states and $r(t)$ is the reference signal), so that the poles of the controller are in $\lambda_{1,2}$=0.5 and the static gain of the closed-loop system is 1.

$$K \;=\; \begin{pmatrix} -2.3717 & 7.5544 \end{pmatrix}$$

The transfer function $Y(z) = W(z)R(z)$ is given by

$$W(z) = C(zI - F + GK)^{-1}GH$$

The static gain is given by

$$W(z)\mid_{z=1} = C(zI - F + GK)^{-1}GH = 1$$

hence

$$H = 1.3948$$