1 Solutions

1.1 Lab problem

a) The relation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \ddot{\phi} \\ \dddot{\phi} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -\frac{1}{T}x_3 + \frac{K}{T}u \end{bmatrix}$$

gives the state space description

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K}{T} \end{bmatrix} u \\ y &= \begin{bmatrix} v & L & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

b) The controllability matrix is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{K}{T} \\ 0 & \frac{K}{T} & -\frac{K}{T^2} \\ \frac{K}{T} & -\frac{K}{T^2} & \frac{K}{T^3} \end{bmatrix}$$

clearly we have $rank(\mathbf{S}) = 3$ (i.e. full rank) and hence a controllable system.

c) The transfer function of PID regulator $F(s)$ could be derived by taking Laplace transform of the time domain expression under the assumption of zero initial conditions:

$$\begin{aligned} U(s) &= K_P E(s) + \frac{K_I}{s} E(s) + K_D E(s) s \\ F(s) &= \frac{U(s)}{E(s)} \\ &= \frac{K_D s^2 + K_P s + K_I}{s} \end{aligned}$$

From the block diagram, the relation in Laplace domain between reference signal $R(s)$ and output signal $Y(s)$ is given by

$$(1 + G(s)F(s))Y(s) = G(s)F(s)R(s)$$

The closed loop system $G_c(s)$ is then defined as

$$\begin{aligned} G_c(s) &= \frac{Y(s)}{R(s)} \\ &= \frac{G(s)F(s)}{1+G(s)F(s)} \\ &= \frac{K(K_D s^2 + K_P s + K_I)}{s^3(Ts+1) + K(K_D s^2 + K_P s + K_I)} \end{aligned}$$

1.2 Multiple choice problem

1. No. The system is just a time delay for three steps. The output has exactly the same signal form as the input but with a time shift. Therefore, a bounded input yields a bounded output.

2. Yes. At any time, the velocity of x is positive which means that the variable can only increase.

3. Yes. Specific solutions of a nonlinear system are analyzed for stability. They might be stable or unstable, depending on where they originate from.

4. Yes. $W(s)|_{s=0} = 1$.

5. Yes. Analog filtering is typically used both at the input and at the output of a discrete controller. Filtering of the input is essential for anti-aliasing, filtering of the output does not let high-frequency content of rectangular control signal shape influence the dynamics of the plant.

6. No, nonlinear systems do not generally preserve the frequency of the input signal in the response. For instance, the static system $y = u^2$ doubles the frequency.

1.3 Problem A

a) Given the equations of the system

$$\frac{dh(t)}{dt} = \frac{f_i(t) - c\sqrt{h(t)}}{S(t)} \tag{1}$$

$$S(t) = \pi(\frac{R}{H}h(t))^2 \tag{2}$$

By replacing (1) into (2) and $f_i(0) = f_{i,0} = 1$ and $\frac{dh(t)}{dt} = 0$, the equilibrium point for the tank level is

$$h(0) = h_0 = \frac{1}{c^2} = 4$$

b) Using Taylor series (until order 1) around the equilibrium point ($f_{i,0}$ and h_0), and assuming $u(t) = f_i(t) - f_{i,0}$ and $x(t) = h(t) - h_0$, the linear space-state representation at the equilibrium point is given by

$$\begin{aligned} \dot{x} &= \left[\frac{1}{\pi}(\frac{H}{R})^2(-0.5ch_0^{-5/2})\right] x + \left[\frac{1}{\pi}(\frac{H}{R})^2 h_0^{-2}\right] u \\ y &= x \end{aligned}$$

By replacing the constant values, the linear space-state representation at the equilibrium point is given by

$$\begin{aligned} \dot{x} &= [-0.0155]\, x + [0.1243]\, u \\ y &= x \end{aligned}$$

c) The output response, $y(t)$ (for $t > 0$) is computed as follows

$$\begin{aligned} y(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-t_0)}Bu(\tau)d\tau \\ y(t) &= 8 - 7.5e^{-0.0155t} \qquad \text{for} \qquad \text{t} > 0 \end{aligned}$$

1.4 Problem B

a) The transfer function of the system is given by

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D \\ G(s) &= \frac{1}{s^2 - 0.01} \end{aligned}$$

b) The characteristic polynomial is given by

$$\Theta(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + 6)$$

$$\Theta(s) = s^3 + 8s^2 + 16s + 24$$

The closed-loop transfer function is given by

$$\begin{aligned} G_{CL}(s) &= \frac{G(s)H(s)}{1 + G(s)H(s)} \\ G_{CL}(s) &= \frac{k_D s^2 + k_P s + k_I}{s^3 + k_D s^2 + (k_P - 0.01)s + k_I} \end{aligned}$$

Matching the coefficients of both polynomials, we have $k_D = 8$, $k_P = 16.01$ and $k_I = 24$.

c) The open loop poles are

$$s_{1,2} = \pm 0.1$$

hence the open loop system is unstable.
The closed-loop poles are

$$(s_{1,2}, s_3) = (-1 \pm j\sqrt{3}, -6)$$

hence the closed loop system is stable.

1.5 Problem C

a) The transition matrix for $t = T$ is

$$\Phi(T) = \begin{pmatrix} 0.8804 & 0.4560 \\ -0.4560 & 0.7892 \end{pmatrix}$$

The sampled system is given by

$$\begin{aligned} x(t+T) &= Fx(t) + Gu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$\begin{aligned} x(t+T) &= \begin{pmatrix} 0.8804 & 0.4560 \\ -0.4560 & 0.7892 \end{pmatrix} x + \begin{pmatrix} 0.1196 \\ 0.4560 \end{pmatrix} u \\ y(t) &= (1 \quad 0)x \end{aligned}$$

Controllability matrix:

$$\mathcal{S} = \begin{pmatrix} 0.1196 & 0.3133 \\ 0.4560 & 0.3054 \end{pmatrix}$$

The system is controllable
Observability matrix:

$$\mathcal{O} = \begin{pmatrix} 1 & 0 \\ 0.8804 & 0.4560 \end{pmatrix}$$

The system is observable

b) By using the Ackermann formula or solving the following equation:

$$\mid (\lambda I - A + LC) \mid = (\lambda - 0.3 + j0.3)(\lambda - 0.3 - j0.3)$$

we have that the estimator gain is given by

$$L = [1.0696 \qquad 0.2660]^T$$

.

c) By using the Ackermann formula or solving the following equation:

$$| (\lambda I - A + BK) |= (\lambda - 0.6 + j0.3)(\lambda - 0.6 - j0.3)$$

we have

$$K = [0.0721 \qquad 1.0107]$$

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The transfer function $Y(z) = W(z)R(z)$ is given by

$$W(z) = C(zI - F + GK)^{-1}GH$$

The static gain of the closed-loop system is given by

$$W(z) \mid_{z=1} = C(zI - F + GK)^{-1}GH = 1$$

hence

$$H = 1.0721$$