Systems and Control
Department of Information Technology
UPPSALA UNIVERSITY
www.it.uu.se/research/syscon

Introduction to computer control systems: Selected exercises for the problem solving sessions

Master program in embedded systems, period 2, 2010

Problem solving session I (Ex1)

1. (Exercise 2.4 from [2])

   Consider a tank with an intermediate wall, that has a hole.

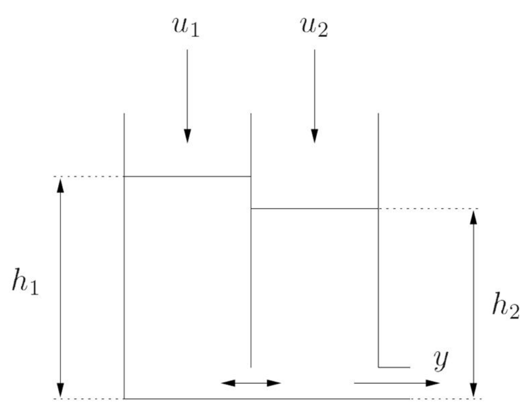

Figure 1: Process diagram

The inflow of water in the left part is denoted by u_1 and the inflow in the right part with u_2. These two flows are inputs. The levels in the tanks are denoted h_1 and h_2. The outflow y is considered to be proportional to the level in the right tank.

$$y(t) = \alpha h_2(t)$$

The flow between the tanks is proportional to the difference in level:

$$f(t) = \beta(h_1(t) - h_2(t))$$

Consider now h_i, u_i and y as deviations from some steady-state values. (They can hence take negative and positive values). Suppose the tank areas are $A_1 = A_2 = 1$. Determine the transfer function from u_1, u_2 to y.

2. (Exercise 2.2 from [2])

   Show that the following system with two inputs

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} x(t) + \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \\ \vdots & \vdots \\ b_n & c_n \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ y(t) &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} x(t)\end{aligned}$$

   has the input-output relation

$$A(p)y(t) = B(p)u_1(t) + C(p)u_2(t)$$

   where

$$\begin{aligned} A(p) &= p^n + a_1 p^{n-1} + \ldots + a_n \\ B(p) &= b_1 p^{n-1} + \ldots + b_n \\ C(p) &= c_1 p^{n-1} + \ldots + c_n \end{aligned}$$

   and p denotes the differentiation operator.

3. (Exercise 2.5 from [2])

   Find a representation of the system

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s^2+s+1)} \end{bmatrix}$$

   on state-space form.

4. (Exercise 2.10 from [1])

   Consider a system of two tanks, where the input signal is the flow to the first tank and the output is the level in the second tank (the output flow of the first tank is the input flow of the second tank). Use of the levels as state variables gives the system

$$\begin{aligned}\frac{dx}{dt} &= \begin{pmatrix} -0.0197 & 0 \\ 0.0178 & -0.0129 \end{pmatrix} x + \begin{pmatrix} 0.0263 \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x\end{aligned}$$

   (a) Sample the system with the sampling period h=12.

   (b) Verify that the pulse-transfer operator for the system is

$$H_0(q) = \frac{0.030q + 0.026}{q^2 - 1.65q + 0.68}$$

5. (Exercise 2.1 from [2]) The following system has been observed

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\
y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)
\end{aligned}
$$

and the following values have been obtained

$$
\begin{array}{ll}
y(1) = 0, & u(1) = 1 \\
y(2) = 1, & u(2) = -1
\end{array}
$$

Determine the state variables values at the time t=3.

References

[1] Karl J. Åström and Björn Wittenmark. *Computer-Controlled Systems.* Prentice Hall, 1997.

[2] Mikael Johansson and Torsten Söderström. *Exercises Control Theory.* Uppsala University and Royal Institute of Technology, 2010.

Answers for Problem solving session I (Ex1)

1. State-space form

$$\begin{aligned}\begin{pmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{pmatrix} &= \begin{pmatrix} -\beta/A_1 & \beta/A_1 \\ \beta/A_2 & -(\alpha+\beta)/A_2 \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} + \begin{pmatrix} 1/A_1 & 0 \\ 0 & 1/A_2 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ y(t) &= \begin{pmatrix} 1 & \alpha \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}\end{aligned}$$

Transfer function (replacing p by s)

$$G(s) = \begin{bmatrix} \frac{\beta\alpha}{s^2+(\alpha+2\beta)s+\beta\alpha} & \frac{\alpha(s+\beta)}{s^2+(\alpha+2\beta)s+\beta\alpha} \end{bmatrix}$$

2. The solution is given in the statement itself.

3. State-space form

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \\ y(t) &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x(t)\end{aligned}$$

4. To be solved in the next session

5. State variables values at the time t=3.

$$x(3) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$