Lecture 2 Mathematical models

- Constants, variables, parameters
- Dynamic models
- State space models
- Impulse response
- Sampling

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Constants, variables, parameters

- Constants: model variables that do not change with time
- System parameters: constants pertaining to system description (lab: the car mass)
- Design parameters: constants that can be selected to give the systems desired properties. (lab: controller parameters)
- Variables (signals): model quantities that vary with time (lab: car position w r t the track)

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Models for dynamic systems

- Dynamic system ⇔ system with memory ⇒ system's output signals depend on previous values of the input signal.
- Mathematic models for dynamic systems can be given by differential or difference equations whose solutions are functions of time.

$$g(y^{(n)}(t), y^{(n-1)}(t), \ldots, y(t), u^{(m)}(t), u^{(m-1)}(t), \ldots, u(t)) = 0$$ input-output (external)form

where *g(.)* is a nonlinear function and

$$y^{(k)}(t) = \frac{d^k}{dt^k} y(t) \qquad y^{(k)}(t) = y(t-k)$$

$$u^{(k)}(t) = \frac{d^k}{dt^k} u(t) \qquad u^{(k)}(t) = u(t-k)$$

continuous time $t \in [0, \infty), t \in (-\infty, \infty)$ — discrete time $t = 0,1,2,\ldots$

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State space description (internal form)

- Input-output form can be a differential equation of high order
- A "simpler" (but equivalent) description is a system of first order differential equations:

$x_i(t), i = 1, \ldots, n$ internal variables (states);

$$x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$ state vector;

$$u = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$ input vector;

$$f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix}$$ vector function;

$$\dot{x} = f(x(t), u(t))$$ state-space description;

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State space models

- Continuous time

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

- Discrete time

$$x(t+1) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

- Linear models

$$f(x,u) = A(t)x + B(t)u$$
$$h(x,u) = C(t)x + D(t)u$$
$$A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}, D \in R^{p \times m}$$

$$x = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad y = \begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} \quad u = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

x – state, n-dimensional vector
y – output, p-dimensional vector
u – input, m -dimensional vector

If *A,B,C,D* are constant matrices then the model is linear and time-invariant

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Solution of the state-space equations for continuous systems

Nonlinear system:

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = h(x(t), u(t))$$

- Set initial conditions $x_0=x(t_0)$
- Set input signal u(t), $t=[t_0,\infty)$
- Integrate numerically to obtain x(t), y(t)

Linear system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

- Set initial conditions $x_0=x(t_0)$
- Set input signal u(t), $t=[t_0,\infty)$
- The solution can be calculated analytically as

$$x(t) = \exp(A(t-t_0))x_0 + \int_{t_0}^{t} \exp(A(t-\theta))Bu(\theta)\, d\theta$$
$$y(t) = Cx(t)$$

- Matrix exponential $\exp(At)$
- Convolution

$$\exp(At) \otimes Bu(t) = \int_{t_0}^{t} \exp(A(t-\theta))Bu(\theta)\, d\theta$$

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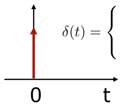
Impulse response: continuous systems

In continuous time, impulse response is the output signal of the system when the input signal is a Dirac delta function.



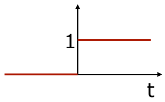
$$\delta(t) = \begin{cases} 0 & t<0 \\ \infty & t=0 \\ 0 & t>0 \end{cases}$$

For impulse response: $t_0 = 0$, $x_0 = 0$, $u(t) = \delta(t)$

$$x(t) = \exp(A(t-t_0))x_0 + \int_{t_0}^{t} \exp(A(t-\theta))Bu(\theta)\, d\theta$$

$$y(t) = Cx(t)$$

Step function (Heaviside function)



Impulse response: $y(t) = C\exp(At)B$

$$\Theta(t) = \begin{cases} 0 & t<0 \\ \text{not defined} & t=0 \\ 1 & t>0 \end{cases}$$

$$\int_{-\infty}^{t} \delta(\theta)\, d\theta = \Theta(t)$$

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Solution of the state-space equations for discrete systems

Nonlinear system:

$$x(t+1) = f(x(t),u(t))$$
$$y(t) = h(x(t),u(t))$$

- Set initial conditions $x_0 = x(t_0)$
- Set input signal u(t), $t = t_0, t_0+1, \ldots$
- Iterate the state equation to obtain x(t), y(t)

Linear system:

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

- Set initial conditions $x_0 = x(t_0)$
- Set input signal u(t), $t = t_0, t_0+1, \ldots$
- The solution can be calculated analytically as

$$x(t) = A^{t-t_0}x_0 + \sum_{i=t_0}^{t-1} A^{t-i-1}Bu(i)$$

$$y(t) = Cx(t)$$

Similarities

$\exp(At) \to A^t$

$\int \to \sum$

Discrete convolution $\quad A^t \otimes Bu(t) = \sum_{i=t_0}^{t-1} A^{t-i-1}Bu(i)$

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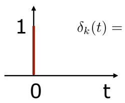
Impulse response: discrete systems

In discrete time, impulse response is the output signal of the system when the input signal is a Kronecker delta function.



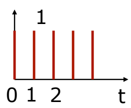
$$\delta_k(t) = \begin{cases} 0 & t<0 \\ 1 & t=0 \\ 0 & t>0 \end{cases}$$

For impulse response: $t_0 = 0$, $x_0 = 0$, $u(t) = \delta_k(t)$

$$x(t) = A^{t-t_0}x_0 + \sum_{i=t_0}^{t-1} A^{t-i-1}Bu(i)$$

$$y(t) = Cx(t)$$

Discrete step function



Impulse response:

$$y(0) = 0$$
$$y(t) = CA^{t-1}B, \ t = 1, 2, \ldots$$

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Sampling

continuous system $\qquad$ discrete system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad y(t) = Cx(t) + Du(t)$$

Sampling time T →

$t = kT, k = 0,1,2,\ldots$

The input signal is assumed to be piece-wise constant

$$u(t) = u(kT), kT \le t < (k+1)T;$$

Sampled system

$$x((k+1)T) = Fx(kT) + Gu(kT)$$
$$y(kT) = Cx(kT) + Du(kT)$$

$$F = e^{AT}, G = \int_0^T e^{A\theta}Bd\theta$$

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Summary

- Systems of differential and difference equations are often used as models for dynamic systems.
- Difference equations can be obtained through sampling (discretization) of differential equations.
- State space models are useful in analysis of dynamic systems, especially for systems of higher order.

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