Lecture 4

- Frequency response
- Poles and zeros
- Stability for linear time invariant systems

Frequency response

Continuous time state space description

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t)$$

$A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{\ell\times n}$
Initial condition x(0)=0

Transfer function:

$$Y(s) = W(s)U(s), W(s) = C(sI - A)^{-1}B$$

Harmonic input signal: $W(s)|_{s=j\omega} = \text{Re } W(j\omega) + j\text{Im } W(j\omega)$

$u(t) = a_0 \sin(\omega_0 t)$ → W(s) → $y(t) = a_0 M(\omega_0)\sin(\omega_0 t + \varphi(\omega_0))$

- Gain characteristics $M(\omega) = \sqrt{\text{Im}^2 W(j\omega) + \text{Re}^2 W(j\omega)}$
- Phase characteristics $\varphi(\omega) = \arctan \frac{\text{Im}W(j\omega)}{\text{Re}W(j\omega)}$

Frequency response

Discrete time state space description

$$x(t+1) = Ax(t) + Bu(t) \qquad y(t) = Cx(t)$$

$A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{\ell\times n}$
Initial condition x(0)=0

Transfer function: $Y(z) = W(z)U(z), W(z) = C(zI - A)^{-1}B$

$W(z)|_{z=e^{j\omega}} = \text{Re } W(e^{j\omega}) + j\text{Im } W(e^{j\omega})$ $\qquad e^{j\omega} = e^{j(\omega+2\pi)}$, $\omega < 2\pi$

$u(t) = a_0 \sin(\omega_0 t)$ → W(z) → $y(t) = a_0 M(\omega_0)\sin(\omega_0 t + \varphi(\omega_0))$

- Gain characteristics $M(\omega) = \sqrt{\text{Im}^2 W(e^{j\omega}) + \text{Re}^2 W(e^{j\omega})}$
- Phase characteristics $\varphi(\omega) = \arctan \frac{\text{Im}W(e^{j\omega})}{\text{Im}W(e^{j\omega})}$

Poles and zeros

The transfer function (matrix-valued) for a state-space model

$$W(s) = C(sI - A)^{-1}B = \frac{C\text{adj}(sI - A)^T B}{\det(sI - A)}$$

Poles are the roots to

$$\det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n = 0$$

i.e. the eigenvalues of A

- For a single-input single output system, the zeros of *W(s)* are the values of *s* such that *W(s)=0.*
- For a multi-input multi-output system, the zeros of *W(s)* are the poles of $W^{-1}(s)$.
- Zeros can be cancelled by poles

Stability

Many different stability notions exist.
Here stability is understood as stability of solutions of system equations with respect to initial conditions.

- Nonlinear systems: stability depends on where in the state space the state vector of the system is.

$\dot{x}(t) = f(x(t))$, $y(t) = h(x(t))$, $x(0) = x_0$ $\qquad$ $x(t+1) = f(x(t))$, $y(t) = h(x(t))$, $x(0) = x_0$

- LTI systems: stability is a system property (defined by systems parameters) and applies notwithstanding the initial conditions

$\dot{x}(t) = Ax(t)$ $\qquad$ $x(t+1) = Ax(t)$ $\qquad$ All parameters are in the matrix *A*

Stability for LTI systems

Initial conditions response

$x(t) = \exp(At)x_0, \ t \in [1, \infty)$ $\qquad$ $x(t) = A^t x_0, \ t = 0, 1, 2, \ldots$

Eigenvalues and eigenvectors of *A*: $A\xi_i = \lambda_i \xi_i, \ i = 1, \ldots, n$

Consider the case of single and distinct eigenvalues:

$$A\begin{bmatrix}\xi_1 & \ldots & \xi_n\end{bmatrix} = \text{diag}(\lambda_1, \ldots, \lambda_n)\begin{bmatrix}\xi_1 & \ldots & \xi_n\end{bmatrix}$$

Introduce the transformation matrix: $T = \begin{bmatrix}\xi_1 & \ldots & \xi_n\end{bmatrix}$

For normalized eigenvectors *T* is unitary, i.e. $T^T T = I$

$T^{-1}AT = T^T AT = \text{diag}(\lambda_1, \ldots, \lambda_n)$

Let $x = Tz$, z –new state vector, $x_0 = Tz_0$

$\dot{z} = \text{diag}(\lambda_1, \ldots, \lambda_n)z$ $\qquad$ $z(t+1) = \text{diag}(\lambda_1, \ldots, \lambda_n)z(t)$

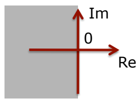
$z(t) = \text{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})z_0$ $\qquad$ $z(t) = \text{diag}(\lambda_1^t, \ldots, \lambda_n^t)z_0$

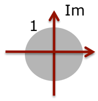
Stability for LTI systems

Continuous system: $z(t) = \mathrm{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) z_0$

Discrete system: $z(t) = \mathrm{diag}(\lambda_1^t, \dots, \lambda_n^t) z_0$

An LTI system is asymptotically stable if and only if all eigenvalues of the matrix A are inside the stability region.





- If an eigenvalue λ_i is outside of the stability region then the system is unstable
- If all egenvalues are inside the stability region or on the stability border and those that are on the stability border are single, then the system is marginally stable. The system output can though be unbounded despite bounded input.

Stability for continuous LTI systems

- LTI – linear time-invariant system (model)
  - Input-output form $a_1, \dots, a_n, b_0, \dots, b_k = \text{const}$

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \dots + a_n y(t) =$$
$$b_0 u^{(k)}(t) + b_1 u^{(k-1)}(t) + \dots + b_k u(t)$$
$$y^{(p)}(t) = \frac{d^p}{dt^p} y(t) \qquad W(s) = \frac{b_0 s^k + b_1 s^{k-1} + \dots + b_k}{s^n + a_1 s^{n-1} + \dots + a_n}$$

  - State space form $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{\ell \times n}$

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad A, B, C = \text{const}$$
$$y(t) = Cx(t) \qquad W(s) = \frac{C\mathrm{adj}(sI - A)^T B}{\det(sI - A)}$$

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_n = 0$$

the characteristic polynomial of *A*, the denominator of *W(s)*

Summary

- Frequency response of a system characterizes its reaction to a harmonic (sinus-like) input and is easiest evaluated from the transfer function of the system
- Zeros of a transfer function specify what frequencies are not reproduced by the system
- Stability is a system property in LTI systems and defined by the eigenvalues of the system matrix in state space form or the denominator of the transfer function, i.e. system poles.