Lecture 5

- Observability and controllability
- Nonlinear systems vs linear
- Stability of equilibria

Observability of LTI systems

Continuous time LTI

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

Discrete time LTI

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t)\end{aligned}$$

- Let $u(t)=0$, $t\geq 0$ and $x(0)=x^*$, $x^*\neq 0$. The state x^* is said to be unobservable if $y(t)=0$ for all $t\geq 0$.
- The system is said to be observable if it lacks unobservable states.
- The system is observable if and only if $\operatorname{rank}\mathcal{O}(A,C)=n$

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\mathcal{O}(A,C)$ is the observability matrix and is the same for the continuous and discrete case

- Any system in observability canonical form is observable

Controllability of LTI systems

- The state x^* is said to be controllable if there is an input that in **finite time** drives the system state vector to x^* from the initial state $x(0)=0$
- The system is controllable if all states are controllable
- The system is controllable if and only if $\operatorname{rank}\mathcal{S}(A,B)=n$

$$\mathcal{S}(A,B) = \begin{bmatrix} B & AB & \dots A^{n-1}B \end{bmatrix}$$

$\mathcal{S}(A,B)$ is the controllability matrix and is the same for the continuous and discrete case

- Any system in controllability canonical form is controllable

Static gain

- The static gain of a system is the finite (asymptotic) value of the system output y when the input u is constant with all elements equal to one.
- Static gain is not defined for unstable and marginally stable systems
- Continuous time:

$$W(s)|_{s=0} = C(sI-A)^{-1}B|_{s=0} = -CA^{-1}B$$

- Discrete time:

$$W(z)|_{z=1} = C(zI-A)^{-1}B|_{z=1} = C(I-A)^{-1}B$$

- Static gain can be established experimentally without a mathematical model

Nonlinear systems versus linear

- Nonlinear system in state-space form

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t))\\ y(t) &= h(x(t), u(t))\end{aligned}$$

- Linear system in state-space form

$$\begin{aligned}f(x(t), u(t)) &= Ax(t) + Bu(t)\\ h(x(t)) &= Cx(t) + Du(t)\end{aligned}$$

- Nonlinear systems versus linear
  - Superposition principle is not valid for nonlinear systems
  - The principle of frequency preservation does not apply to nonlinear systems
  - Linear models can be seen as approximations of nonlinear systems) linearization

Equilibria

The point (x_0, u_0) is an equilibrium for

$$\dot{x}(t) = f(x(t), u(t))$$

if $f(x_0, u_0) = 0$

Fix u_0=*const* and let $x(0)=x_0$ yielding $x^*(t)= x_0$

- Stability is not a system property in nonlinear systems. Stability of solutions and equilibria is rather investigated.
- The equilibrium (x_0, u_0) is stable, unstable or asymptotically stable if the solution $x^*(t)$ has the corresponding property
- For LTI-system $(0,0)$ is always an equilibrium

$$Ax_0 = Bu_0$$

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Stability for equilibria via linearization

Assume that the nonlinear system

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

is described in vicinity of the equilibrium (x_0, u_0) by the linear system

$$\dot{z}(t) = Az(t) + Bv(t);\ \ A = \frac{\partial f}{\partial x},\ \ B = \frac{\partial f}{\partial u}$$

where $z = x - x_0$ and $v = u - u_0$.

- If all eigenvalues of A have strictly negative real part, then (x_0, u_0) is an asymptotically stable equilibrium.
- If any of the egenvalues of A has strictly positive real part then (x_0, u_0) is an unstable equilibrium
- If none of the eigenvalues of A has positive real part but there are eigenvalues on the imaginary axis then the equilibrium can be either stable or unstable.

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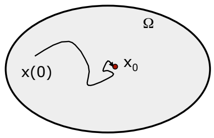
Stability of equilibria via linearization

- To establish stability properties of equilibria there is no need in studying the solutions of the nonlinear system in question. Stability properties are defined by the stability properties of the linearized system and linear theory is sufficient.
- If an equilibrium is asymptotically stable then it is surrounded by an attraction domain. All solutions that start within the attraction domain converge to the equilibrium. The size of an attraction domain is typically difficult to estimate.



$$x(0) \in \Omega \Rightarrow x(t) \to x_0, t \to \infty$$

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Summary

- Controllability and observability are important prerequisites for controller design
- Static gain is a system characteristics that can be measured experimentally
- Nonlinear systems are much more difficult to analyze than linear ones.
- Linearization of nonlinear systems can be used to investigate stability of equilibria. In some cases though, it is necessary to analyze the nonlinear system, anyway.