Informationsteknologi

Lecture 6

- Pole placement in state-space form
- Observers
- Feedback from reconstructed states

State feedback controller

Continuous time LTI: $\dot{x}(t) = Ax(t) + Bu(t)$

Discrete time LTI: $x(t+1) = Ax(t) + Bu(t)$

State feedback: $u(t) = -Kx(t)$

Closed-loop continuous time: $\dot{x} = (A - BK)x(t)$

Closed-loop discrete time: $x(t+1) = (A - BK)x(t)$

The aim of control is to to drive the state vector to zero asymptotically from any initial state x_0.

Discrete time solution: $x(t) = \exp\left((A - BK)t\right)x_0,\quad t \in [1, \infty)$

Discrete time solution: $x(t) = (A - BK)^t x_0,\quad t = 0, 1, 2, \ldots$

- Controller design: select *K* so that *A-BK* corresponds to an asymptotically stable system.
- Stability of the closed-loop system is defined by the eigenvalues of *A-BK*

Pole placement by state feedback

- The feedback gain *K* can be selected so that *A-BK* attains arbitrary pre-defined eigenvalues if and only if the controllability matrix has full rank $\operatorname{rank} \mathcal{S}(A,B) = n$

$$\mathcal{S}(A,B) = \begin{bmatrix} B & AB & \ldots A^{n-1}B \end{bmatrix}$$

- The eigenvalues of *A-BK* are the poles of the closed-loop system, i.e. the roots of the characteristic polynomial

$$\alpha_c(A - BK) = \det(sI - A + BK) = 0$$

- The design procedure that places the eigenvalues of *A-BK* at a set of desired locations is called pole placement
  - Ackermann's formula is used for SISO systems

$$K = \begin{bmatrix} 0 & 0 & \ldots 1 \end{bmatrix} \mathcal{S}(A,B)^{-1} \alpha_c(A)$$

  - More advanced algorithms for MIMO system
- If the pair *(A,B)* is in controllability canonical form, then the pair *(A-BK,B)* is also in controllability canonical form.

State observers

- The state feedback controller assumes the state vector to be available for measurement.
- The state vector is composed of internal variables that are not necessary measured and their estimates are therefore are sought.
- System model is available and can be used for estimation of non-measurable internal variables
- Continuous time: $\dot{x} = Ax + Bu$, $y = Cx$
- Discrete time: $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$
- With known *A* and *B*, a mathematical model can be fed with the same control signal as the system itself

$$\dot{\hat{x}} = A\hat{x} + Bu;\ \hat{x}(0) = \hat{x}_0;\ \hat{x} - \text{state estimate}$$

State observers, contd.

- State estimation error for the model

$$\dot{\hat{x}} = A\hat{x} + Bu \qquad \dot{e} = Ae$$
$$e(t) = x(t) - \hat{x}(t) \qquad e(t) = \exp(At)(x_0 - \hat{x}_0)$$

- When *A* has all the eigenvalues in the left half-plane $e(t) \to 0, t \to \infty$
- What to do when *A* does not give fast enough convergence or unstable?
- Use feedback! *e(t)* cannot be measured but *Ce(t)* can be.
  $Ce(t) = Cx(t) - C\hat{x}(t) = y - \hat{y};\ \hat{y} = C\hat{x}$
  $\dot{e} = Ae - LCe$
- Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \qquad \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right)$$

Design of state observers

The feedback gain *L* is selected to make the state estimation error to go to zero asymptotically

$$\dot{e} = (A - LC)e \qquad e(t+1) = (A - LC)e(t)$$

Design problem: choose *L* so that all eigenvalues of *A-LC* are within the stability domain.

Note: any square matrix *A* has the same eigenvalues as A^T.

$$\det(sI - A + LC) = \det(sI - A^T + C^T L^T) = 0$$

- Controller: If *(A,B)* is a controllable pair then the eigenvalues of *A-BK* can be assigned arbitrarily by the choice of *K*.
- Observer: If *(A,C)* is an observable pair then the eigenvalues of *A-LC* can be assigned arbitrarily by the choice of *L*.
- If *(A,B)* is an observable pair then (A^T,B^T) is an observable pair.
- Observer design can therefore be reduced to controller design by pole placement in A^T-C^TL^T that gives the value of *L*.

Feedback from reconstructed states

Plant in state space form:

$\dot{x} = Ax + Bu$ $\qquad x(t+1) = Ax(t) + Bu(t)$

$y = Cx$ $\qquad y(t) = Cx(t)$

Observer to estimate the states from the output

$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$ $\qquad \hat{x}(t+1) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right)$

Feedback stabilizing controller from estimated states:

$$u(t) = -K\hat{x}(t)$$

Feedback stabilizing controller with reference signal

$$u(t) = -K\hat{x}(t) + r(t)$$

The closed-loop system:

$\dot{x} = Ax + Bu$ $\qquad x(t+1) = Ax(t) + Bu(t)$

$\dot{e} = (A - LC)e$ $\qquad e(t+1) = (A - LC)e(t)$

Feedback from reconstructed states, contd

Closed-loop system in state-space form

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

- Transfer function *Y(s)=W(s)R(s)* $\quad W(s) = C(sI - A + BK)^{-1}B$
- The state estimation error is not controllable but observable
- Characteristic polynomial of the closed-loop system

$$\det(sI - A + BK)\det(sI - A + LC) = 0$$

- The closed-loop system is stable whenever the following systems are stable:

$\dot{e} = (A - LC)e$ $\qquad \dot{x} = (A - BK)x$

- The dynamics of the observer should be faster than the dynamics of the state feedback controller to alleviate the impact of the state estimation error on the controller performance

Summary

- For a controllable LTI system with measurable state vector its dynamics can be arbitrarily changed by a static state feedback.
- For an observable system, all the state variables can be estimated from measurements of the input and the output by an observer.
- For an observable and controllable system, its dynamics can be arbitrarily changed by feeding back the estimates of the state variables provided by a state observer.
- The observer and controller design in an observer-based controller should be performed so that the observer dynamics is much faster than the controller dynamics.