Introduction to Computer Control Systems
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Maths Tutorial

Contents

1 Complex numbers

1.1 Definitions

A complex number is a number of the form $a + jb$, where a and b are real numbers and j is an imaginary unit, satisfying $j^2 = -1$. For example, $3 + j2$ is a complex number.

Complex plane
A complex number can be viewed as a point or position vector in a two-dimensional Cartesian coordinate system called the *complex plane*. The complex number is plotted using the real part as the horizontal component, and the imaginary part as vertical (see Figure 1).

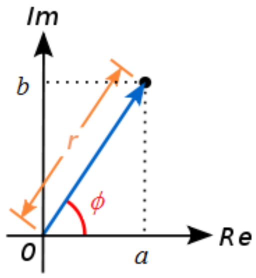

Figure 1: Representation in the complex plane

Magnitude (r) **and Phase** (ϕ).
As a vector in the complex plane, an alternative way of defining a complex number is to use its magnitude and angle. Applying basic trigonometry to the vector in Figure 1, the magnitude and phase are given by

$$r = \sqrt{a^2 + b^2}\ ;\ \phi = \arctan\left(\frac{b}{a}\right) \tag{1.1}$$

It follows that the complex number $z = a + jb$ can be expressed as

$$z = r\cos\phi + jr\sin\phi = re^{j\phi} \tag{1.2}$$

The phase is the angle with respect to the positive real axis, and is expressed in radians. Then, in general we have

$$\phi = \begin{cases} \arctan\left(\frac{b}{a}\right) & \text{if } a > 0 \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan\left(\frac{b}{a}\right) - \pi & \text{if } a < 0 \text{ and } b < 0 \end{cases}$$

Any increase by an integer multiple of 2π will given the same angle.

Complex conjugate
The complex conjugate of a complex number is another complex number with imaginary parts of opposite sign. This is, the conjugate of the complex number $z = a + jb$ is $\bar{z} = a - jb$.

1.2 Basic operations

Addition and substraction
Complex numbers are added/substracted by adding/substracting the real and imaginary parts of the summands. This is,

$$(a + jb) \pm (c + jd) = (a \pm c) + j(b \pm d) \tag{1.3}$$

Multiplication
Similar to the rules for multiplying two binomials, and since $j^2 = -1$, the multiplication of two complex numbers $(a + jb)$ and $(c + jd)$ gives

$$(a + jb)(c + jd) = (ac - bd) + j(ad + bc) \tag{1.4}$$

Division

The division can be obtained by using the multiplication and the complex conjugate properties. For example, the division of two complex numbers $(a + jb)$ and $(c + jd)$, where $c \neq 0$ and $d \neq 0$, gives

$$\frac{a+jb}{c+jd} = \frac{(a+jb)(c-jd)}{(c+jd)(c-jd)} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2} = \frac{(ac+bd)}{c^2+d^2} + j\frac{(bc-ad)}{c^2+d^2} \tag{1.5}$$

2 Matrix algebra

2.1 Definitions

Rows and columns

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in *rows* and *columns*, as shown in Figure 2. Here each element of the matrix is denoted by two subscripts (i, j). For example, $a_{2,1}$ represents the element at the second row and first column of the matrix.

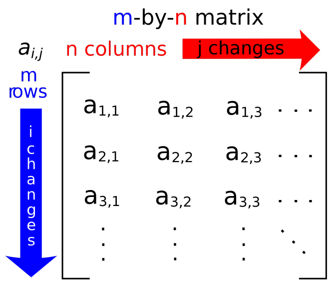

Figure 2: Elements of a matrix

Size

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix.

Matrix transpose

The transpose of a matrix A is another matrix, denoted A^T, created by any of the following equivalent actions:

- write the rows of A as the columns of A^T
- write the columns of A as the rows of A^T

Then, if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. For example:

$$\begin{bmatrix} 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad ; \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Identity matrix

The identity matrix, also called *unit matrix*, of size $n \times n$ has ones on the main diagonal and zeros elsewhere. It can be denoted by I_n or simply by I. Then we have,

$$I_1 = \begin{bmatrix} 1 \end{bmatrix} \ , \ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , \dots , \ I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{2.1}$$

If A is a $n \times n$ square matrix, it is a property of matrix multiplication (see **Matrix multiplication** for details about this operation) that

$$I_n A = A I_n = A \tag{2.2}$$

Matrix Rank

In linear algebra, the rank of a matrix A, commonly denoted as $\text{rank}(A)$, is the size of the largest collection of linearly independent columns of A (the column rank) or the size of the largest collection of linearly independent rows of A (the row rank). Then, for a $m \times n$ matrix A we have

$$\text{row rank of } A \leq m \quad ; \quad \text{column rank of } A \leq n \tag{2.3}$$

For every matrix, the column rank is equal to the row rank. Therefore, there is no reason to distinguish between row rank and column rank; the common value is simply called the rank of the matrix. It follows that

$$\text{rank}(A) \leq \min(m, n) \tag{2.4}$$

where $\min(m, n)$ denotes the smaller of the two numbers m and n (or their common value if $m = n$). For example, the rank of a 3×5 matrix can be no more than 3, and the rank of a 4×2 matrix can be no more than 2.

Another example, the matrix A given by $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ has rank 2: the first two rows are linearly independent, so the rank is at least 2, but all three rows are linearly dependent (the first is equal to the sum of the second and third) so the rank must be less than 3.

For example, the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \end{bmatrix}$ has rank 1: any pair of columns in linearly dependent.

A common approach to finding the rank of a matrix is to reduce it to a simpler form by elementary row operations. Let's see this approach with an example. assume again the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ This matrix can be put in a reduce form by using the following elementary row operations:

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{r_2 = 2r_1 + r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{r_3 = -3r_1 + r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow{r_3 = r_2 + r_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 = -2r_2 + r_1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

where, for example, $\xrightarrow{r_2 = 2r_1 + r_2}$ means that row 2 is replaced by 2 times row 1 plus row 2. The final matrix has two non-zero rows and thus the rank of matrix A is 2.

2.2 Basic Operations

Matrix addition and substraction
Two matrices may be added or substracted only if they have the same dimension (same number of rows and columns).

For example, if A is a 2×2 matrix and B is a 2×2 matrix, the addition (or substraction) gives a 2×2 matrix, as follows

$$A \pm B = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \pm \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 & a_2 \pm b_2 \\ a_3 \pm b_3 & a_4 \pm b_4 \end{bmatrix} \tag{2.5}$$

Scalar-matrix multiplication
The multiplication of a matrix A with a scalar α gives another matrix αA of the same size as A. For example,

$$\alpha A = \alpha \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha a_2 \\ \alpha a_3 & \alpha a_4 \end{bmatrix} \tag{2.6}$$

Matrix-vector multiplication
To calculate the multiplication between a matrix A and a vector v, first we need to view the vector as a column matrix. We define the matrix-vector multiplication only if the number of columns in A equals the number of rows in v. This is, if A is an $m \times n$ matrix (matrix with n columns), then the product Av is defined for $n \times 1$ vectors v.

For example, if we have a 2×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}$ and a vector $v = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then the result is a 2×1 vector

$$Av = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} (a_1)(b_1) + (a_2)(b_2) + (a_3)(b_3) \\ (a_4)(b_1) + (a_5)(b_2) + (a_6)(b_3) \end{bmatrix} \tag{2.7}$$

Matrix-matrix multiplication
Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their matrix product AB is the $m \times p$ matrix whose entries are given by dot product of the corresponding row of A and the corresponding column of B.

For example, if A is a 2×3 matrix and B is a 3×2 matrix, the multiplication gives a 2×2 matrix, as follows

$$AB = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1b_1 + a_2b_3 + a_3b_5 & a_1b_2 + a_2b_4 + a_3b_6 \\ a_4b_1 + a_5b_3 + a_6b_5 & a_4b_2 + a_5b_4 + a_6b_6 \end{bmatrix} \tag{2.8}$$

Determinants
Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A, also denoted by $\det(A)$ or $|A|$, is defined to be

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \tag{2.9}$$

Given a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, the determinant of A is defined to be

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + b_1c_2a_3 + a_2b_3c_1 - [a_3b_2c_1 + b_1a_2c_3 + a_1b_3c_2] \tag{2.10}$$

A $k \times k$ determinant can be expanded by using the (i, j) minors (see **Minors and Cofactors**) to obtain

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \dots & a_{2k} \\ \vdots & \ddots & \vdots \\ a_{k2} & \dots & a_{2k} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k3} & \dots & a_{kk} \end{vmatrix} + \dots \pm a_{1k} \begin{vmatrix} a_{21} & \dots & a_{2(k-1)} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{k(k-1)} \end{vmatrix} \tag{2.11}$$

Minors and Cofactors
A minor of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns. Then, the (i, j) minor of a square matrix, denoted as M_{ij}, is the determinant of the sub-matrix formed by deleting the i-th row and j-th column. The (i, j) cofactor, denoted as C_{ij}, is given by

$$C_{ij} = (-1)^{(i+j)} M_{ij} \tag{2.12}$$

For example, given a matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, the minor M_{23} and the cofactor C_{23} of A are

$$M_{23} = \det \begin{bmatrix} a_1 & a_2 & \bullet \\ \bullet & \bullet & \bullet \\ c_1 & c_2 & \bullet \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} = a_1 c_2 - (c_1 a_2)$$

$$C_{23} = (-1)^{(2+3)} M_{23} = -\left[a_1 c_2 - (c_1 a_2)\right]$$

2.3 Matrix Inverse

The inverse of a square matrix A, is a matrix A^{-1} such that $AA^{-1} = I$, where I is the identity matrix. A square matrix A has an inverse if and only if the determinant $|A| \neq 0$. A matrix possessing an inverse is called *nonsingular* or *invertible*.

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2.13}$$

For a 3×3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, the matrix inverse is

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \tag{2.14}$$

where C_{ij} is the (i, j) cofactor (see expression 2.12).

2.4 Summary of important matrix properties

Table 1: Some matrix properties

Not commutative	$AB \neq BA$
Distributive over matrix addition	Left distributivity: $A(B + C) = AB + AC$
	Right distributivity: $(A + B)C = AC + BC$
Scalar multiplication	$\alpha\,(AB) = (\alpha A)\,B$ and $(AB)\,\alpha = A\,(B\alpha)$
Transpose	$(AB)^T = B^T A^T$
Identity element*	$AI = IA = A$
Inverse matrix*	$AA^{-1} = A^{-1}A = I$
	$(AB)^{-1} = B^{-1}A^{-1}$
Determinants*	$\det(AB) = \det(A)\det(B)$
Power of matrices*	$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$
	$A^0 = I$
	$(\alpha A)^k = \alpha^k A^k$
	$\det(A^k) = \det(A)^k$

α is a scalar, k is a positive integer. (*) Square matrices only.

2.5 Eigenvalues and eigenvector

An eigenvector of a square matrix A is a non-zero vector v that, when the matrix multiplies v, yields the same as when some scalar multiples v. This is

$$Av = \lambda v \tag{2.15}$$

The number λ is called the eigenvalue of A corresponding to v. The eigenvalue equation for a matrix A is

$$Av - \lambda v = 0 \implies (A - \lambda I)\, v = 0 \tag{2.16}$$

It is a fundamental result of linear algebra that an equation $Bv = 0$ has a non-zero solution v if and only if, the determinant $|B| = 0$. Then, it follows that the **eigenvalues** of A in expression (2.16) are precisely the real numbers λ that satisfy the equation

$$\det(A - \lambda I) = 0 \tag{2.17}$$

where $\det(A - \lambda I)$ is called the **characteristic polynomial** of A.

For example, the eigenvalues of $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$ are

$$\det \begin{bmatrix} 2-\lambda & -4 \\ -1 & -1-\lambda \end{bmatrix} = 0 \longrightarrow (2-\lambda)(-1-\lambda) - (-4)(-1) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

which has two solutions: $\lambda_1 = 3$ and $\lambda_2 = -2$ called the eigenvalues of A.

Let's find the eigenvectors for the corresponding eigenvalues. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

<u>For $\lambda_1 = 3$ we have:</u>

$$(A - \lambda_1 I)v = 0 \longrightarrow (A - 3I) = \begin{bmatrix} 2-3 & -4 \\ -1 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} -v_1 - 4v_2 = 0 \\ -v_1 - 4v_2 = 0 \end{matrix}$$

If we let $v_2 = t$, then $v_1 = -4t$. All eigenvectors corresponding to $\lambda_1 = 3$ are multiples of $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

<u>For $\lambda_2 = -2$ we have:</u>

$$(A - \lambda_1 I)v = 0 \longrightarrow (A - (-2)I) = \begin{bmatrix} 2-(-2) & -4 \\ -1 & -1-(-2) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{matrix} 4v_1 - 4v_2 = 0 \\ -v_1 + v_2 = 0 \end{matrix}$$

If we let $v_2 = t$, then $v_1 = t$. All eigenvectors corresponding to $\lambda_2 = -2$ are multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3 The derivative

3.1 Definition

The derivative is a fundamental tool of calculus. For example, the derivative of the position of a moving object with respect to time is the object's velocity: this measures how quickly the position of the object changes when time is advanced. The derivative measures the instantaneous rate of change of the function.

The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value. In this way, the derivative at a point of a function $f(x)$ of a single variable is the slope of the tangent line to the graph of the function at that point (see Figure 3). Using mathematical notation, the derivative of a function $y = f(x)$, also denoted as $f'(x)$ or dy/dx, is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.1}$$

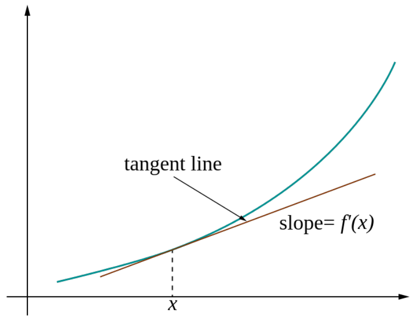

Figure 3: The tangent line at $(x, f(x))$

and is defined as the limit of the average rate of change in the function as the length of the interval on which the average is computed tends to zero. When the derivative is taken n times, it is denoted as $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$.

3.2 Derivative of basic functions

Most derivative computations eventually require taking the derivative of some common functions. A list of some of the most frequently used functions of a single real variable and their derivatives is presented below.

Table 2: Derivative of basic functions

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
a	0	$\sin(x)$	$\cos(x)$
x^a	ax^{a-1}	$\cos(x)$	$-\sin(x)$
e^x	e^x	$\tan(x)$	$\sec^2(x)$
a^x	$a^x \ln(a)$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\ln(x)$	$\frac{1}{x}$	$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$
$\log_a(x)$	$\frac{1}{x\ln(a)}$	$\arctan(x)$	$\frac{1}{1+x^2}$

A table of basic rules for differentiating functions is given below

Table 3: Basic rules for differentiating functions

Case	Derivative
$(\alpha f \pm \beta g)$	$\alpha f' \pm \beta g'$
$(fg)'$	$f'g + fg'$
$\left(\frac{f}{g}\right)'$	$\frac{f'g - fg'}{g^2}$
$(f(g(x)))'$	$f'(g(x))g'(x)$

3.3 Examples

Example 1. Calculate the derivative of $f(x) = x^4 + \sin(x^2) - \ln(x)e^x + 7$

Using the derivative of basic functions given in Table 2 and the properties given in Table 3, we have

$$\begin{aligned} f'(x) &= 4x^{(4-1)} + \left[\sin(x^2)\right]' - [\ln(x)e^x]' + 0 \\ &= 4x^3 + \left[\cos(x^2)2x\right] - \left[\frac{1}{x}e^x + \ln(x)e^x\right] \\ &= 2x\left[2x^2 + \cos(x^2)\right] - e^x\left[\frac{1}{x} + \ln(x)\right] \end{aligned}$$

4 Laplace Transform

4.1 Definition

The Laplace transform is an integral transform particularly useful in solving linear ordinary differential equations (ODEs). The Laplace transform $\mathscr{L}$ of $f(t)$ is defined by

$$\mathscr{L}\left\{f(t)\right\} = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{4.1}$$

where $f(t)$ is a function defined for $t \geq 0$.

4.2 Examples

Some examples of the Laplace transform are presented below.

Example 1: Exponential function. Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ ae^{-bt} & \text{for } t \geq 0 \end{cases}$$

where a and b are constants. The Laplace transformation of this exponential function can be obtained as follows:

$$\mathscr{L}\left[ae^{-bt}\right] = \int_0^\infty ae^{-bt}e^{-st}dt = a\int_0^\infty e^{-(b+s)t}dt = \frac{a}{s+b}$$

Example 2: Step function. Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ a & \text{for } t > 0 \end{cases}$$

where a is a constant. Note that this is a special case of the exponential function ae^{-bt} where $b = 0$. The Laplace transformation is given by:

$$\mathscr{L}\left[a\right] = \int_0^\infty ae^{-st}dt = \frac{a}{s}$$

Example 3: Ramp function. Consider the function

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ at & \text{for } t \geq 0 \end{cases}$$

where a is a constant. The Laplace transformation is given by:

$$\mathscr{L}\left[at\right] = \int_0^\infty ate^{-st}dt = a\left[t\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt\right] = a\left[\frac{e^{-st}}{-s^2}\bigg|_0^\infty\right] = \frac{a}{s^2}$$

A list of several important Laplace transforms is given in Table 4.

Table 4: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	e^{ct}	$F(s-c)$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

A list of important properties of the Laplace transforms is given in Table 5.

Table 5: Properties of Laplace Transforms

$\mathscr{L}[af(t)] = aF(s)$	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$	$\mathscr{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$	$\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s); n = 1,2,3,...$
$\mathscr{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sF(s) - f'(0)$	$\mathscr{L}\left[f(\frac{t}{a})\right] = aF(as)$
$\mathscr{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]\Big\|_{t=0}$	$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s)$
$\mathscr{L}\left[e^{-at}f(t)\right] = F(s+a)$	

5 Ordinary Differential Equations (ODEs)

5.1 Definition

An ODE is an equality involving a function and its derivatives. An ODE of order n is an equation of the form

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0 \tag{5.1}$$

where y is a function of x, $y' = dy/dx$ is the first derivative with respect to x, and $y^{(n)} = d^n y/dx^n$ is the n^{th} derivative with respect to x.

An ODE of order n is said to be **linear** if it is in the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y^{'} + a_0(x)y = Q(x) \tag{5.2}$$

In general, an n^{th}-order ODE has n linearly independent solutions. Furthermore, any linear combination of linearly independent functions is also a solution.

5.2 Solving Linear ODEs using Laplace Transforms

It is possible to use Laplace transform to solve linear ODEs. The procedure is explained with some examples:

Example 1. Consider the ODE

$$y'' - 5y' + 6y = 0, \text{ where } y(t) \text{ and } y(0) = 2,\ y'(0) = 2$$

Let $Y(s) = \mathscr{L}[y(t)]$. Instead of solving directly for $y(t)$, we derive a new equation for $Y(s)$. Once we find $Y(s)$, we inverse the transform to get $y(t)$.

Therefore, the Laplace transform of the ODE is

$$\mathscr{L}[y'' - 5y' + 6y] = \mathscr{L}[0]$$
$$\mathscr{L}[y''] - 5\mathscr{L}[y'] + 6\mathscr{L}[y] = 0$$

Applying the properties of the Laplace transform (see Table 5), we have

$$\left\{s^2\mathscr{L}[y] - sy(0) - y'(0)\right\} - 5\left\{s\mathscr{L}[y] - y(0)\right\} + 6\mathscr{L}[y] = 0$$

Since $y(0) = 2$ and $y'(0) = 2$ we get

$$\left\{s^2 Y(s) - 2s - 2\right\} - 5\left\{sY(s) - 2\right\} + 6Y(s) = 0$$

Now, the idea is to use some common transforms we already saw in Table 4. Therefore, solving for $Y(s)$ and applying partial fractions we have

$$Y(s) = \frac{2s - 8}{s^2 - 5s + 6} = \frac{4}{s-2} + \frac{-2}{s-3}$$

Then, using the Table 4 we can obtain the inverse transforms, giving

$$y(t) = \mathscr{L}^{-1}[Y(s)] = 4\mathscr{L}^{-1}\left[\frac{1}{s-2}\right] - 2\mathscr{L}^{-1}\left[\frac{1}{s-3}\right]$$
$$= 4e^{2t} - 2e^{3t} \quad , \quad \text{for } t \geq 0$$

Example 2. Consider the ODE

$$y'' + 2y' + 5y = 3, \text{ where } y(t) \text{ and } y(0) = 0,\ y'(0) = 0$$

Since $y(0) = 0$, and $y'(0) = 0$, the Laplace transform of the ODE becomes

$$\left\{s^2 Y(s)\right\} + 2\left\{sY(s)\right\} + 5\left\{Y(s)\right\} = \frac{3}{s}$$

Solving for $Y(s)$ and applying partial fractions to get similar transforms as shown in Table 4, we have

$$Y(s) = \frac{3}{s\left(s^2+2s+5\right)} = \frac{3}{5}\left[\frac{1}{s} - \frac{s+2}{s^2+2s+5}\right] = \frac{3}{5s} - \frac{3}{10}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{3}{5}\left[\frac{s+1}{(s+1)^2+2^2}\right]$$

The inverse Laplace transform becomes

$$\begin{aligned} y(t) = \mathscr{L}^{-1}[Y(s)] &= \frac{3}{5}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathscr{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] - \frac{3}{5}\mathscr{L}^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] \\ &= \frac{3}{5} - \frac{3}{10}e^{-t}\sin(2t) - \frac{3}{5}e^{-t}\cos(2t) \quad , \quad \text{for } t \geq 0 \end{aligned}$$

6 Taylor series

6.1 Definitions

The idea is to obtain a linear mathematical function from a nonlinear function. In this case, assume that the variables deviate only slightly from some operating condition. Consider a function whose input is $x(t)$ and output is $y(t)$, and the relationship between $y(t)$ and $u(t)$ is given by

$$y = f(x) \tag{6.1}$$

If the normal operating condition corresponds to $x = a, y = b$, then $y = f(x)$ may be expanded into a Taylor series about this point as follows:

$$\begin{aligned} y &= f(x) \\ &= \underbrace{\underbrace{f(a) + f'(a)(x-a)}_{1^{st} \text{ order expansion}} + \frac{1}{2!} f''(a)(x-a)^2}_{2^{nd} \text{ order expansion}} + \ldots + \frac{1}{n!} f^n(a)(x-a)^n + \ldots \end{aligned} \tag{6.2}$$

where $f'(a), f''(a), \ldots, f^n(a), \ldots$ are the derivatives evaluated at $x = a$. Note that we call **first order** and **second order** Taylor expansion to a Taylor series calculated until the first and second derivative, respectively.

The linearization procedure is based on the expansion of the nonlinear function into a Taylor series about the operating point and the retention of only the linear term. Because we neglect high-order terms of Taylor series expansion, these neglected terms must be small enough; that is, the variable deviate only slightly from the operating condition.

6.2 Examples

Some examples of using Taylor expansion to linearize some functions are presented below.

Example 1. Find the first order Taylor expansion of $f(x) = e^x$ about $x = a$

It means that we want to express $f(x) = e^x$ around $x = a$ as

$$f(x) = f(a) + f'(a)(x-a)$$

Since $f'(x) = (e^x)' = e^x$, then the first order Taylor expansion is

$$e^x = e^a + e^a(x-a)$$

Example 2. Find the first order Taylor expansion of $f(x) = x\sin(x)$ about $x = a$

Note that $f(x)$ involves the multiplication of two functions: x and $\sin(x)$. Then, to obtain the first derivative we follow the product rule for derivatives (see Table 3). It gives that $f'(x) = (x\sin(x))' = \sin(x) + x\cos(x)$. Then, the first order Taylor expansion is

$$x\sin(x) = a\sin(a) + (\sin(a) + a\cos(a))\,(x-a)$$