Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2};(a^2\neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at};(n=1,2,3...)$	$\frac{1}{(s+a)^n}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}\left[af(t)\right]=aF(s)$	$\mathcal{L}\left[tf(t)\right]=-\frac{dF(s)}{ds}$
$\mathcal{L}\left[f_1(t)\pm f_2(t)\right]=F_1(s)\pm F_2(s)$	$\mathcal{L}\left[t^2f(t)\right]=\frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0)$	$\mathcal{L}\left[t^nf(t)\right]=(-1)^n\frac{d^n}{ds^n}F(s);n=1,2,3,...$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right]=s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f(\frac{t}{a})\right]=aF(as)$
$\mathcal{L}\left[\int f(t)dt\right]=\frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)dt\right]\big|_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right]=F_1(s)F_2(s)$
$\mathcal{L}\left[e^{-at}f(t)\right]=F(s+a)$	

Matrix exponential

$$e^{At}\triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$$

Open-loop and sensitivity functions

$$G_o(s)=G(s)F_y(s),\qquad S(s)=\frac{1}{1+G_o(s)},\qquad T(s)=1-S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

-

$$S = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At} x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

Feedback control structures

General linear feedback form:

$$U(s) = F_r(s)R(s) - F_y(s)Y(s)$$

We can write the following control structures in Laplace form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K \left(\frac{\tau_D s + 1}{\beta \tau_D s + 1} \right) \left(\frac{\tau_I s + 1}{\tau_I s + \gamma} \right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$F_r(s) = \left(1 - L(sI - A + KC + BL)^{-1}B \right) \ell_0$$
$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$x(k+1) = Fx(k) + Gu(k)$$
$$y(k) = Hx(k)$$

where

$$F = e^{AT}$$
$$G = \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B$$
$$H = C$$