Intro. Computer Control Systems: F2

Transfer function, poles and stability

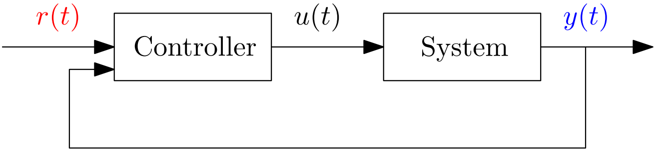
Dave Zachariah

Dept. Information Technology, Div. Systems and Control

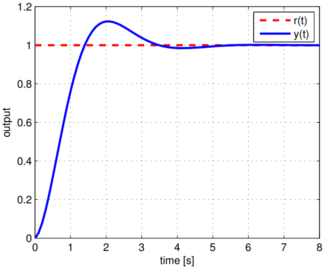
dave.zachariah@it.uu.se

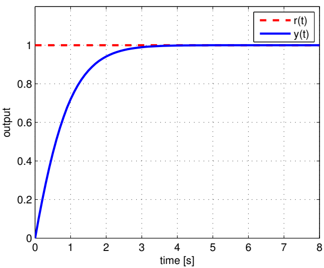
F1: Quiz!

Suppose a control system



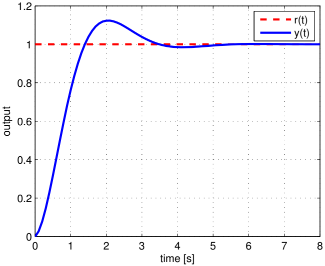
comes with two different settings (a) and (b).

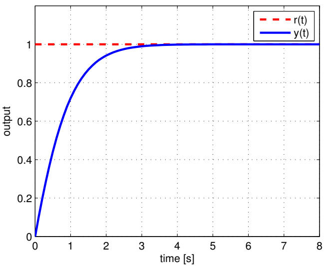




dave.zachariah@it.uu.se

F1: Quiz!





1) Which setting of the controller is intuitively better?
   - a Setting to the left ↑
   - b Setting to the right ↑
   - c They are equally good ↓

dave.zachariah@it.uu.se

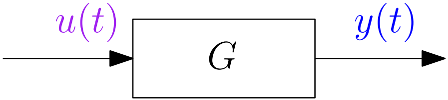
Linear system models

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Linear system models

Linear time-invariant models are useful and sufficiently accurate in many control applications.



Linear ODE:s are *one possible* input-output description, i.e. of G:

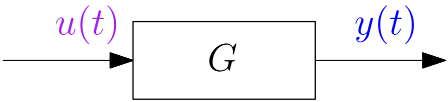
$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0 \frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

with initial conditions.

dave.zachariah@it.uu.se

Linear system models

Linear time-invariant models are useful and sufficiently accurate in many control applications.



Linear ODE:s are *one possible* input-output description, i.e. of G:

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

with initial conditions.

Rarely practical in analysis or design for control!

dave.zachariah@it.uu.se

Laplace transform

Used as tool to *solve* and *analyze* linear ODE:s

- **Notation**:

$$y(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \mathcal{L}\left[y(t)\right] = Y(s)$$

dave.zachariah@it.uu.se

Laplace transform

Used as tool to *solve* and *analyze* linear ODE:s

- **Notation:**

$$y(t) \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad \mathcal{L}\left[y(t)\right] = Y(s)$$

- **Definition:**

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt, \quad s \in \mathbb{C}$$

Inverse transform:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2\pi i}\int_{\mathbb{C}} Y(s)e^{st}ds, \quad s \in \mathbb{C}$$

Note that s and $Y(s)$ are complex-valued!

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Important properties

linearity: $y(t) = \alpha x(t) + \beta z(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) = \alpha X(s) + \beta Z(s)$

derivatives: $\frac{dy}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0)$

$$\frac{d^2y}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - \dot{y}(0)$$

$$\vdots$$

integral: $\int_0^t y(\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}Y(s)$

convolution: $\int_0^t x(\tau)z(t-\tau)d\tau \overset{\mathcal{L}}{\longleftrightarrow} X(s)Z(s)$

final-value thm.*: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Solving linear ODE with $\mathcal{L}$

Example: solve output $y(t)$

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \qquad u(t), y(0), \dot{y}(0) \quad \text{given}$$

dave.zachariah@it.uu.se

Solving linear ODE with $\mathcal{L}$

Example: solve output $y(t)$

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \qquad u(t), y(0), \dot{y}(0) \quad \text{given}$$

$\rightarrow$ Laplace transform

$$\begin{aligned}\text{LHS} &= s^2Y(s) - sy(0) - \dot{y}(0) + 2\,(sY(s) - y(0)) + 3Y(s) \\ &= (s^2 + 2s + 3)Y(s) - (s+2)y(0) - \dot{y}(0) \\ \text{RHS} &= 4\,(sU(s) - u(0)) + 5U(s) = (4s+5)U(s) - 4u(0)\end{aligned}$$

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Solving linear ODE with $\mathcal{L}$

Example: solve output $y(t)$

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \qquad u(t), y(0), \dot{y}(0) \quad \text{given}$$

$\rightarrow$ Laplace transform

$$\begin{aligned} \text{LHS} &= s^2Y(s) - sy(0) - \dot{y}(0) + 2\left(sY(s) - y(0)\right) + 3Y(s) \\ &= (s^2 + 2s + 3)Y(s) - (s+2)y(0) - \dot{y}(0) \\ \text{RHS} &= 4\left(sU(s) - u(0)\right) + 5U(s) = (4s+5)U(s) - 4u(0) \end{aligned}$$

$\Rightarrow$ Set LHS = RHS and solve for $Y(s)$:

$$Y(s) = \frac{4s+5}{s^2+2s+3}U(s) + \frac{s+2}{s^2+2s+3}y(0) + \frac{1}{s^2+2s+3}(\dot{y}(0) - 4u(0))$$

dave.zachariah@it.uu.se

Solving linear ODE with $\mathcal{L}$

Example: solve output $y(t)$

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = 4\frac{d}{dt}u + 5u, \qquad u(t), y(0), \dot{y}(0) \quad \text{given}$$

$\rightarrow$ Laplace transform

$$\begin{aligned} \text{LHS} &= s^2Y(s) - sy(0) - \dot{y}(0) + 2\left(sY(s) - y(0)\right) + 3Y(s) \\ &= (s^2 + 2s + 3)Y(s) - (s+2)y(0) - \dot{y}(0) \\ \text{RHS} &= 4\left(sU(s) - u(0)\right) + 5U(s) = (4s+5)U(s) - 4u(0) \end{aligned}$$

$\Rightarrow$ Set LHS $=$ RHS and solve for $Y(s)$:

$$Y(s) = \frac{4s+5}{s^2+2s+3}U(s) + \frac{s+2}{s^2+2s+3}y(0) + \frac{1}{s^2+2s+3}(\dot{y}(0) - 4u(0))$$

$\Rightarrow$ Compute $y(t) = \mathcal{L}^{-1}[y(t)]$ using $\mathcal{L}^{-1}$-transform (table)

Given: $U(s) = \mathcal{L}[u(t)]$, $u(0)$, $y(0)$ and $\dot{y}(0)$

dave.zachariah@it.uu.se

Transfer function and impulse response

dave.zachariah@it.uu.se

Transfer function $G(s)$

- Assuming initial values are zero $y(0) = \dot{y}(0) = \cdots = 0$ and $u(0) = \dot{u}(0) = \cdots = 0$. Effect of input u on output y:

$$Y(s) = \underbrace{\frac{4s+5}{s^2+2s+3}}_{G(s)} U(s),$$

where $G(s)$ is the system transfer function $u \to y$.

dave.zachariah@it.uu.se

Transfer function $G(s)$

- Assuming initial values are zero $y(0) = \dot{y}(0) = \cdots = 0$ and $u(0) = \dot{u}(0) = \cdots = 0$. Effect of input u on output y:

$$Y(s) = \underbrace{\frac{4s+5}{s^2+2s+3}}_{G(s)} U(s),$$

  where $G(s)$ is the system transfer function $u \to y$.
- More generally,

$$\boxed{Y(s) = G(s)U(s)}$$

  is a model of the *relation* between the system input u and output y.

dave.zachariah@it.uu.se

Transfer function

- A system described by the linear ODE

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

  with initial values 0.

- Laplace transform of both sides:

$$(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0 s^m + \cdots + b_{m-1}s + b_m)U(s)$$

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Transfer function

- A system described by the linear ODE

$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

  with initial values 0.

- Laplace transform of both sides:

$$(s^n + \cdots + a_{n-1}s + a_n)Y(s) = (b_0 s^m + \cdots + b_{m-1}s + b_m)U(s)$$

- System transfer function is a *rational* function:

$$\boxed{G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

Note that s and $G(s)$ are complex-valued!

dave.zachariah@it.uu.se

Weighting function/impulse response

A system $Y(s) = G(s)U(s)$ (at rest $t = 0$) yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \int_0^t g(\tau)u(t-\tau)d\tau,$$

i.e. a *convolution* between $u(t)$ and

$$\boxed{g(t) = \mathcal{L}^{-1}[G(s)]}$$

denoted the system weighting function.

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Weighting function/impulse response

A system $Y(s) = G(s)U(s)$ (at rest $t = 0$) yields

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \int_0^t g(\tau)u(t-\tau)d\tau,$$

i.e. a *convolution* between $u(t)$ and

$$\boxed{g(t) = \mathcal{L}^{-1}[G(s)]}$$

denoted the system weighting function.

Suppose input $u(t)=\delta(t)$ = (Dirac)pulse, then output

$$\boxed{y(t) = \int_0^t g(\tau)\delta(t-\tau)d\tau = g(t).}$$

Hence $g(t)$ is called the system impulse response.

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Poles, zeros and stability

dave.zachariah@it.uu.se

Poles and zeros

Characterizing system behaviour

System with transfer function $G(s)$

- **Zeros:** s' is a *zero*, if $G(s') = 0$.
- **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.

dave.zachariah@it.uu.se

Poles and zeros
Characterizing system behaviour

System with transfer function $G(s)$

- **Zeros:** s' is a *zero*, if $G(s') = 0$.
- **Poles:** s' is a *pole*, if $G(s')$ is a singularity, that is, $G(s') = \pm\infty$.
- If $G(s) = \frac{B(s)}{A(s)}$ is a *rational function*
  - zeros = the roots to $B(s) = 0$,
  - poles = the roots to $A(s) = 0$.

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

- Assume system $Y(s) = G(s)U(s)$, where $G(s) = \frac{B(s)}{A(s)}$.
- We want

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

where $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$.

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

- Assume system $Y(s) = G(s)U(s)$, where $G(s) = \frac{B(s)}{A(s)}$.
- We want

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

where $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$.

- Denominator always factorize with roots/poles:

$$\begin{aligned} A(s) &= s^n + a_1 s^{n-1} + \cdots + a_n \\ &= (s+\sigma_1)(s+\sigma_2)\cdots((s+\sigma_j)^2 + \omega_j^2)\cdots \end{aligned}$$

where poles are either

  - real-valued: $-\sigma_1, \ldots$
  - complex-conjugated: $-\sigma_j \pm i\omega_j, \ldots$

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

- Now insert $A(s)$ and use partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_1}{s+\sigma_1} + \cdots + \frac{B_j(s)}{(s+\sigma_j)^2 + \omega_j^2} + \cdots$$

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

- Now insert $A(s)$ and use partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_1}{s+\sigma_1} + \cdots + \frac{B_j(s)}{(s+\sigma_j)^2 + \omega_j^2} + \cdots$$

- Impulse response $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$ using table:

$$g(t) = \beta_1 e^{-\sigma_1 t} + \cdots + b_j e^{-\sigma_j t} \sin(\omega_j t + \varphi_j) + \cdots$$

and system output

$$y(t) = \int_0^t g(\tau) u(t-\tau) d\tau$$

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

- Now insert $A(s)$ and use partial-fraction decomposition

$$G(s) = \frac{B(s)}{A(s)} = \frac{\beta_1}{s + \sigma_1} + \cdots + \frac{B_j(s)}{(s + \sigma_j)^2 + \omega_j^2} + \cdots$$

- Impulse response $g(\tau) = \mathcal{L}^{-1}[B(s)/A(s)]$ using table:

$$g(t) = \beta_1 e^{-\sigma_1 t} + \cdots + b_j e^{-\sigma_j t} \sin(\omega_j t + \varphi_j) + \cdots$$

and system output

$$y(t) = \int_0^t g(\tau) u(t - \tau) d\tau$$

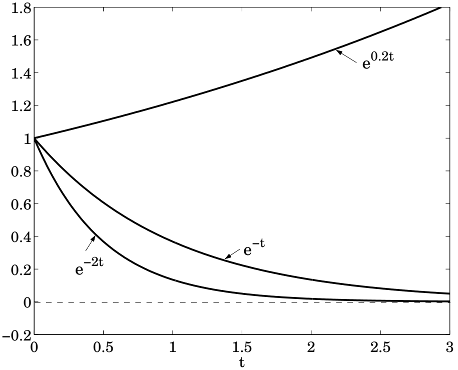
Output as linear combination of exponential functions

dave.zachariah@it.uu.se

Poles and solution to linear ODE:s

Characterizing system behaviour

Output as linear combination of exponential functions



Real-parts of poles ($-\sigma$) play an important role

Stability

Characterizing system behaviour

Definition:

A system $Y(s) = G(s)U(s)$ is input-output stable if all bounded inputs $u(t)$ yield a bounded output $y(t)$.

Bounded signal $x(t)$ means $\Leftrightarrow |x(t)| \leq K$ for some K.

[Board: bounded impulse response + real-part of poles]

dave.zachariah@it.uu.se

Stability

Characterizing system behaviour

Definition:

A system $Y(s) = G(s)U(s)$ is input-output stable if all bounded inputs $u(t)$ yield a bounded output $y(t)$.

Bounded signal $x(t)$ means $\Leftrightarrow |x(t)| \leq K$ for some K.

[Board: bounded impulse response + real-part of poles]

Result:

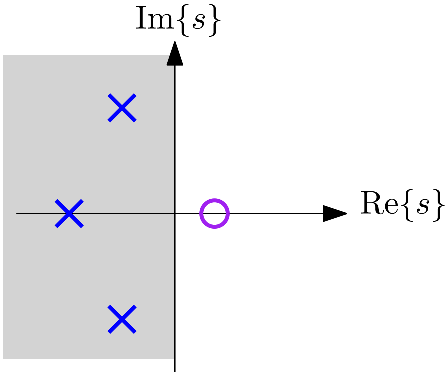
Assume $G(s) = B(s)/A(s)$ with poles $s = p_1, p_2, \ldots, p_n$ (and order of denominator $\geq$ numerator)

$$Y(s) = G(s)U(s) \text{ input-output stable } \Leftrightarrow \operatorname{Re}\{p_i\} < 0$$

dave.zachariah@it.uu.se

Graphical representation of poles and zeros

Characterizing system behaviour



$G(s) = \frac{B(s)}{A(s)}$ stable $\Leftrightarrow$ poles lie in left-halfplane

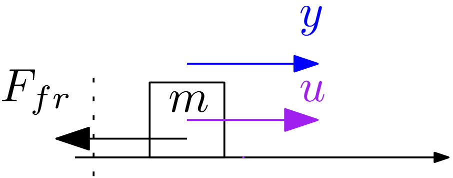
dave.zachariah@it.uu.se

Examples

dave.zachariah@it.uu.se

Build intuition from simple systems

Ex. #1: Vehicle in motion



Figur : Force $u(t)$ and velocity $y(t)$.

Standard form:

$$\frac{d}{dt}y + \left(\frac{C}{m}\right) y = \left(\frac{1}{m}\right) u$$

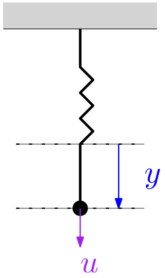
[Board: poles]

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Build intuition from simple systems

Ex. #2: Damper



Figur : Force $u(t)$ and position $y(t)$.

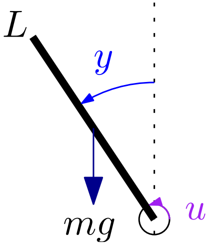
Standard form:

$$\frac{d^2}{dt^2}y + \left(\frac{K}{m}\right) y = \left(\frac{1}{m}\right) u$$

[Board: poles]

Build intuition from simple systems

Ex. #3: Inverted pendulum pendel



Figur : Torque $u(t)$ and angle $y(t)$.

Standard form (around $y \approx 0$):

$$\frac{d^2}{dt^2} y - \left(\frac{3g}{2L} \right) y = \left(\frac{3}{mL^2} \right) u$$

[Board: poles]

dave.zachariah@it.uu.se

Summary and recap

- Transfer functions as a system description
- Poles and zeros
- (Bounded) input-output stability

dave.zachariah@it.uu.se