



Intro. Computer Control Systems: F3

Time response, feedback and PID-control

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F2: Quiz!

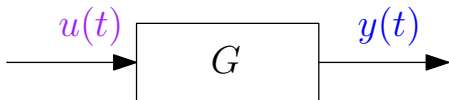
- 1) Systems with **impulse response** $g(t) = \mathcal{L}^{-1}[G(s)]$, where $G(s)$ is rational, are all
- a Products of sinusoidal functions \uparrow
 - b Linear combinations of exponential functions \uparrow
 - c Stable \downarrow



Poles and time responses

Poles and time responses

Step response

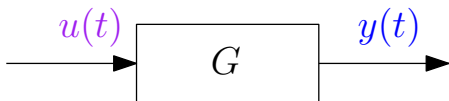


Model $Y(s) = G(s)U(s)$ with transfer function

$$G(s) = \frac{b_0 s^m + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}.$$

Poles and time responses

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Suppose input is a step:

$$u(t) = \begin{cases} u_0 \text{ (const.)} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \xleftrightarrow{\mathcal{L}} \quad U(s) = \frac{u_0}{s}$$

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Study **step response** $y(t)$ of the system

Poles and time responses

Poles distance from the origin \leftrightarrow quickness

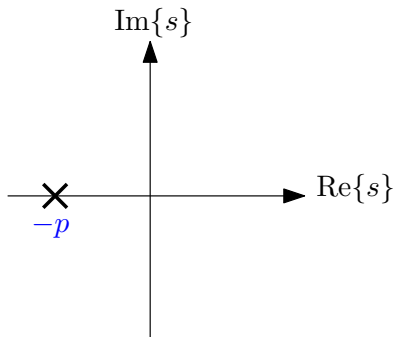
Ex. 1st-order system:

$$G(s) = [\text{typ. form}] = \frac{p}{s + p}$$

Pole at:

$$s = -p$$

Pole-zero diagram:



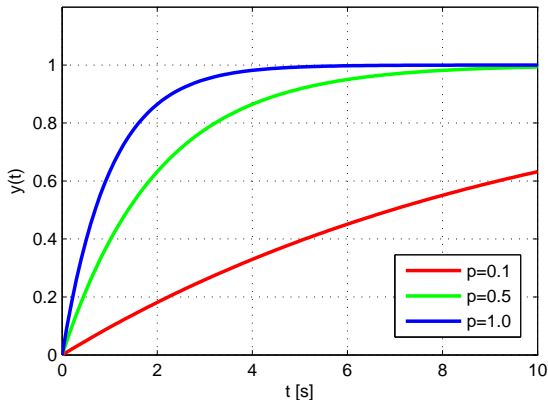
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Step response:



Poles and time responses

Complex-conjugated poles \leftrightarrow system **oscillations**

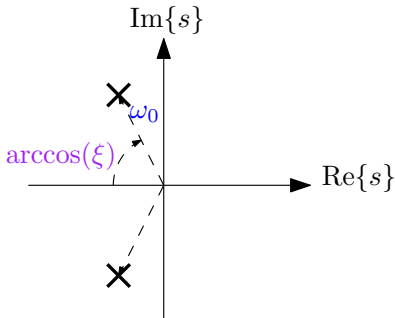
Ex. 2nd-order system:

$$G(s) = [\text{alt. form}] = \frac{\omega_0^2}{s^2 + 2\xi\omega_0 s + \omega_0^2}$$

Poles at:

$$s = -\omega_0\xi \pm i\omega_0\sqrt{1 - \xi^2}$$

Pole-zero diagram:



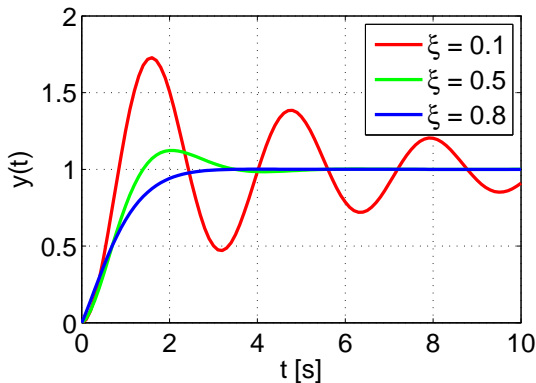
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Step response:



Poles and time responses

Pole that is closest to the origin \leftrightarrow **slowest** time constant

Ex.: 3rd-order system

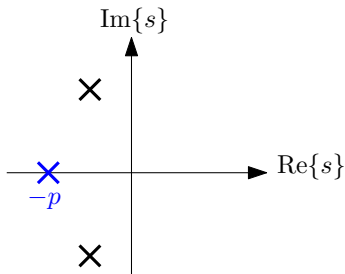
$$G(s) = \frac{p}{s + p} \cdot \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

with $\omega_0 = 1$ and $\zeta = \sqrt{0.5}$.

Poles at:

$$s = -p \quad \text{och} \quad s = -\omega_0\xi \pm i\omega_0\sqrt{1 - \xi^2}$$

Pole-zero diagram:



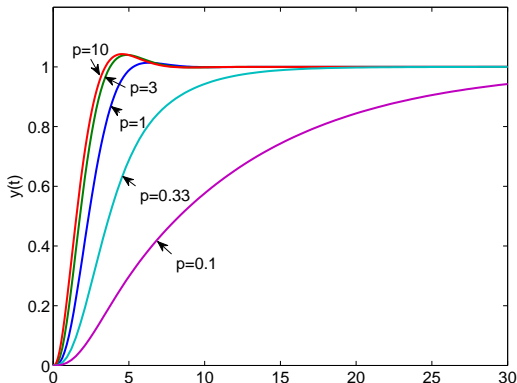
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Static gain of a system

Assume *stable* system $Y(s) = G(s)U(s)$ with

$$u(t) = \begin{cases} u_0 \text{ (const.)} & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \xleftrightarrow{\mathcal{L}} \quad U(s) = \frac{u_0}{s}$$

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Final value of step response $y(t)$ can be computed using **final value theorem**:

$$y_f = \lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} sG(s) \frac{u_0}{s} =$$

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The system **static gain** is therefore $G(0)$

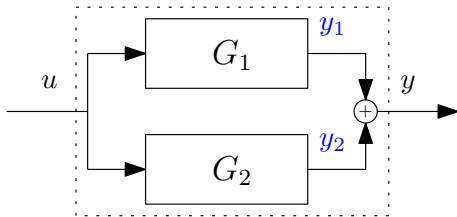


Connected and feedback systems

Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Parallel-connected systems



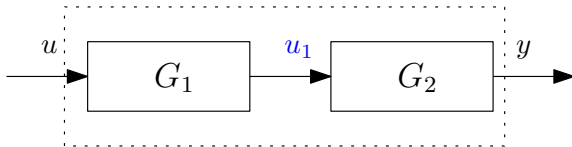
With added help signals:

$$\begin{aligned}
 Y(s) &= Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)U(s) \\
 &= (G_1(s) + G_2(s))U(s)
 \end{aligned}$$

Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Serial-connected systems



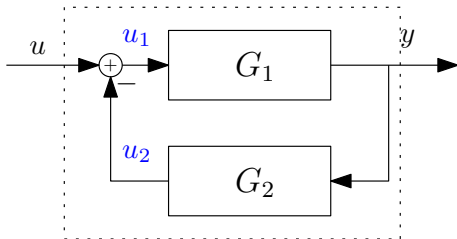
With added help signals:

$$\begin{aligned}
 Y(s) &= G_2(s)U_1(s) = G_2(s)(G_1(s)U(s)) \\
 &= G_2(s)G_1(s)U(s)
 \end{aligned}$$

Connected and feedback systems

Transfer function obtained using Laplace + added help signals

Ex.: Feedback systems

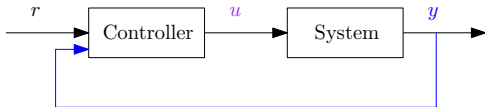


[Board: derive transfer function $u \rightarrow y$]

Feedback control based on error signal

PID-control

Feedback control:



Simple control strategy: Use the **control error**

$$e(t) \triangleq r(t) - y(t)$$

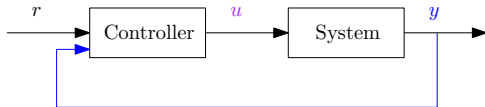
[Board: intuition from temperature control]

to determine input $u(t)$

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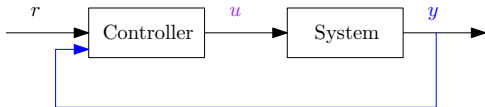
to determine input $u(t)$ based on:

- ▶ *current* error: $\propto e(t)$ (Proportional)

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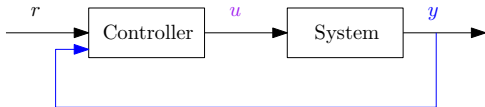
to determine input $u(t)$ based on:

- ▶ *current* error: $\propto e(t)$ (P)roportional
- ▶ *past* error: $\propto \int_{\tau=0}^t e(\tau) d\tau$ (I)ntegral

Feedback control based on error signal

PID-control

Feedback control:



Simple control strategy: Use the **control error**

$$e(t) \triangleq r(t) - y(t)$$

[Board: intuition from temperature control]

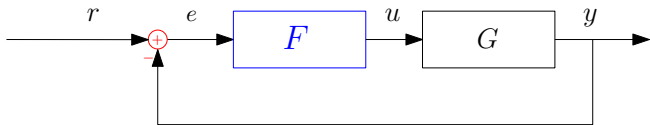
to determine input $u(t)$ based on:

- ▶ *current* error: $\propto e(t)$ (**P**roportional)
- ▶ *past* error: $\propto \int_{\tau=0}^t e(\tau) d\tau$ (**I**ntegral)
- ▶ *change* in error: $\propto \dot{e}(t)$ (**D**erivative)

Ideal PID-controller

Controller with user parameters:

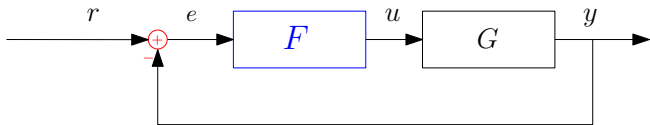
$$u(t) = \underbrace{K_p e(t)}_P + K_i \underbrace{\int_{\tau=0}^t e(\tau) d\tau}_I + \underbrace{K_d \dot{e}(t)}_D$$



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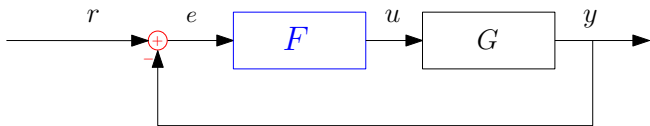


Laplace domain

Ideal PID-controller

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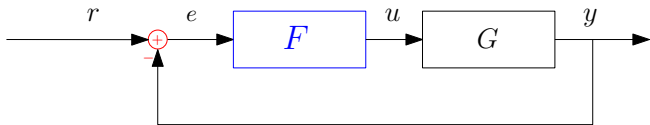
Laplace domain:

$$\begin{aligned}
 U(s) &= K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) \\
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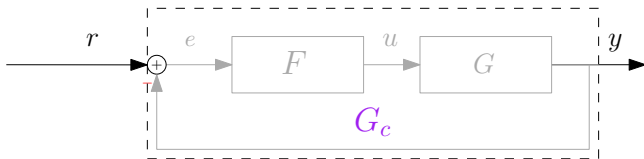
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Analysis of simple feedback control

Simple feedback control system

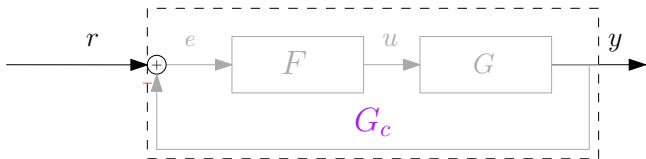
Closed-loop system from reference to output



- ▶ Closed-loop system from r to y : $G_c(s)$
- ▶ Open-loop system from e to y : $G_o(s) = G(s)F(s)$

Simple feedback control system

Closed-loop system from reference to output



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[Board: derive closed-loop system G_c]

Note: We can design the poles of G_c !



Accuracy: stationary control error

Using a step as reference signal r

Assume stable $G_c(s)$ with reference (step):

$$r(t) = \begin{cases} r_0 & \text{for } t \geq 0, \\ 0 & \text{for } t < 0, \end{cases} \quad \xleftrightarrow{\mathcal{L}} \quad R(s) = \frac{r_0}{s}$$

Final value of error $e(t)$

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Final value of error $e(t) \xleftrightarrow{\mathcal{L}} E(s) = R(s) - Y(s)$:

$$\begin{aligned} e_f &= \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} s \frac{1}{1 + G_0(s)} \frac{r_0}{s} \\ &= \lim_{s \rightarrow 0} \frac{r_0}{1 + G(s)F(s)} \end{aligned}$$

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Result:

Stationary error e_f approaches **0** if $G(0)F(0) = \infty$. (E.g. when $F(s)$ contains $\frac{1}{s}$, i.e. *integration*.)



Summary and recap

- ▶ Relation between poles and system time-response
- ▶ The transfer function for connected and feedback systems
- ▶ Simple feedback control and ideal PID-controller
- ▶ Control accuracy: stationary error