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# Intro. Computer Control Systems: F6

## Bode plot, design in frequency domain, Nyquist contour, minimum phase

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# F5: Quiz!

1) General feedback control
   - a Leads to unstable closed-loop systems $\uparrow$
   - b Leads to more design freedom $\uparrow$
   - c Leads to non-minimum phase systems $\downarrow$


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# F5: Quiz!

1) General feedback control
   - a Leads to unstable closed-loop systems ↑
   - b Leads to more design freedom ↑
   - c Leads to non-minimum phase systems ↓
2) For linear time-invariant systems a sinusoidal input yields
   - a A sinusoidal output ↑
   - b A exponentially declining output ↑
   - c A stable output ↓



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# Frequency response of system

# Properties in the frequency domain

How does a (closed-loop) system respond to oscillating signals?



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# Properties in the frequency domain

How does a (closed-loop) system respond to oscillating signals?

- Input signals decomposed into sum of cosine- and sine signals:

$$r(t) = \frac{1}{2\pi} \int R(i\omega) e^{i\omega t} d\omega,$$

where

$$e^{i\omega t} = \cos(\omega t) + i \sin(\omega t) \quad \text{and} \quad \omega = 2\pi f \quad \text{(frequency)}$$


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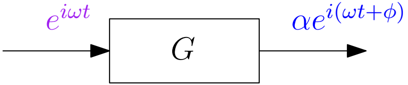
- $e^{i\omega t}$ is an eigen-function to linear time-invariant systems:



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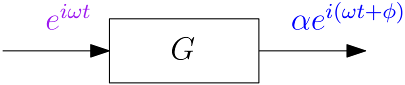
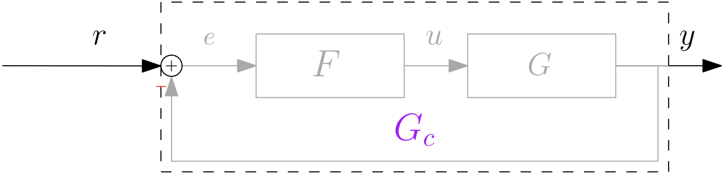



- $\Rightarrow$ System output $y(t)$ is reweighted sum of the *input* cosine- and sine signals!

# Frequency response
## Example of (closed-loop) system




$$Y(s) = G_c(s)R(s)$$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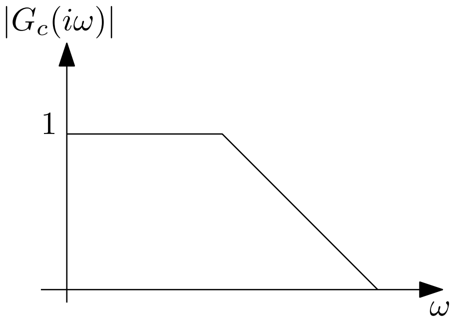
# Frequency response
## Example of (closed-loop) system

Frequency reponse: $G_c(s)\Big|_{s=i\omega} = G_c(i\omega)$

Example: Magnitude curve of frequency response





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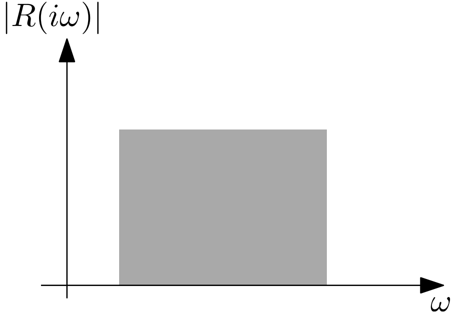
# Frequency response

## Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

$$r(t) = \frac{1}{2\pi} \int R(i\omega) e^{i\omega t} d\omega.$$

Example: Frequency content of signal




Note that the plot is symmetric $|X(i\omega)| = |X(-i\omega)|$



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# Frequency response

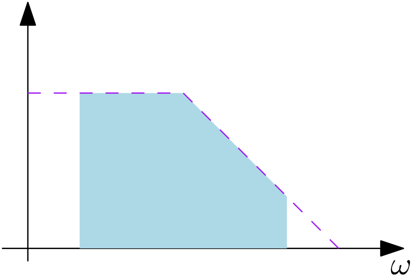
## Example of (closed-loop) system

Signal as a weighted sum of cosine- and sine signals:

$$y(t) = \frac{1}{2\pi} \int Y(i\omega) e^{i\omega t} d\omega.$$

Example: Frequency content of output

$$|Y(i\omega)| = |G_c(i\omega)||R(i\omega)|$$



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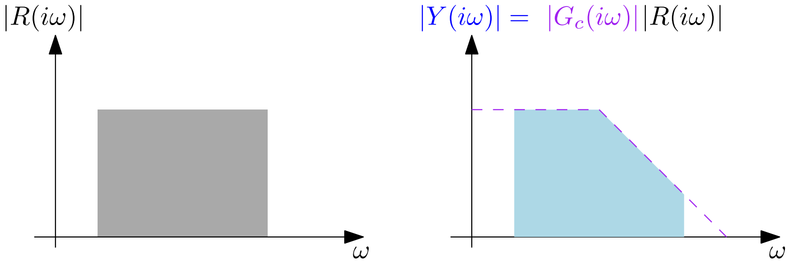
# Frequency response

## Example of (closed-loop) system




Cf. "sine-in sine-out"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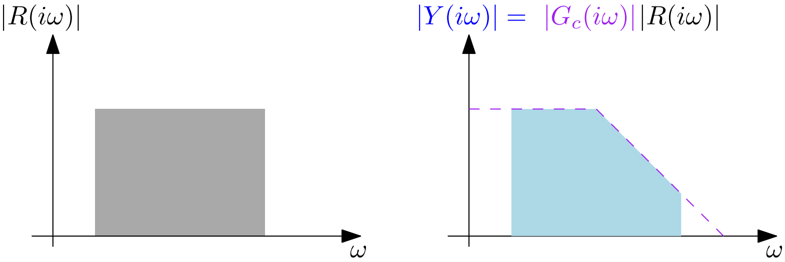
# Frequency response

## Example of (closed-loop) system




Goal: $y(t) \approx r(t) \Leftrightarrow Y(i\omega) \approx R(i\omega)$

- Closed-loop system and frequency response:

$$Y(i\omega) = G_c(i\omega)R(i\omega) = |G_c(i\omega)|e^{i\arg\{G_c(i\omega)\}}R(i\omega)$$

- *Ideal:* Magnitude $|G_c(i\omega)| \approx 1$ and phase $\arg\{G_c(i\omega)\} \approx 0$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# Sketching the frequency response



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# Poles/zeros and frequency response

## Bode plot: recipe for magnitude curve

Re-write *standard form* using the poles and zeros:

$$G(s) = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} \;=\; K_0 \frac{(s + z_1) \cdots (s + z_m)}{(s + p_1) \cdots (s + p_n)}$$



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# Poles/zeros and frequency response

## Bode plot: recipe for magnitude curve

Re-write on *modified form*:

$$G(s) = \cdots = K_0 \frac{(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = K \frac{(1+\frac{s}{z_1})\cdots(1+\frac{s}{z_{m'}})}{s^q(1+\frac{s}{p_1})\cdots(1+\frac{s}{p_{n'}})}$$



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# Poles/zeros and frequency response

## Bode plot: recipe for magnitude curve

System on *modified* form:

$$G(s) = K \frac{(1+\frac{s}{z_1})\cdots}{s^q(1+\frac{s}{p_1})\cdots}$$

*Logarithm* of |frequency response|:

$$\log_{10}|G(i\omega)| = \log_{10}|K| - q\log_{10}|\omega| + \log_{10}|1+\frac{i\omega}{z_1}| + \cdots - \log_{10}|1+\frac{i\omega}{p_1}| - \cdots$$

*Intuition:* Each term is turned on as $\omega$ increases

- when $\omega \ll |z_k|, |p_k|$, the term is $\approx 0$
- when $\omega \gg |z_k|, |p_k|$, the term increases/decreases with $\omega$


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# Poles/zeros and frequency response

## Bode plot: recipe for magnitude curve

### Recipe for sketching magnitude curve

$$\boxed{\log_{10} |G(i\omega)|}$$

1. Compute all $z_k$ and $p_k$
2. Sort $|z_k|$ or $|p_k|$ by distance to the origin.
3. Evaluate $\log |G(i\omega)|$ at the first $|z_1|$ or $|p_1|$ after the origin.
4. Plot the curve along $\omega \to \infty$:
   - Each zero $\omega \gg |z_k|$ increases the slope by $+1$.
   - Each pole $\omega \gg |p_k|$ decreases the slope by $-1$.
   - Complex-conjugated poles give resonance peak at $\omega \approx |p_k|$ if $\zeta \ll 1$.



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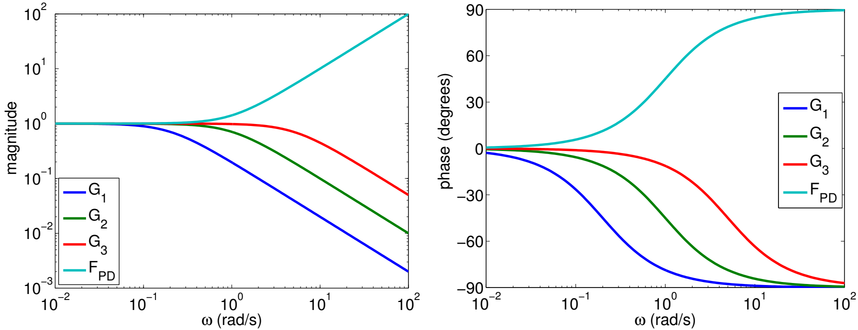
# Bode plot
## Example

1:st order system with simple poles.

Examples:

$$G_1(s) = \frac{0.2}{s+0.2}, \qquad G_2(s) = \frac{1}{s+1},$$

$$G_3(s) = \frac{5}{s+5}, \qquad (F_{PD}(s) = 1 + s)$$



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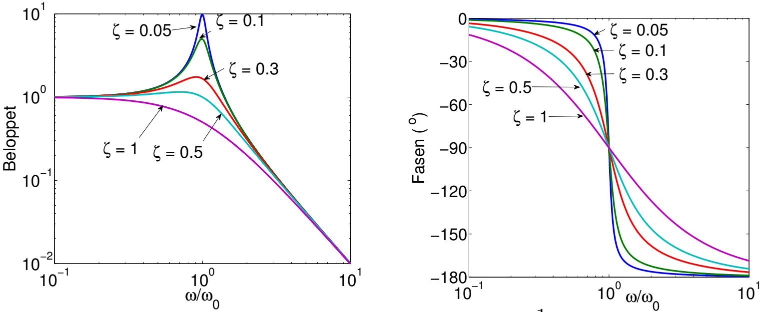
# Bode plot
## Example

2nd order system with complex conjugated poles:

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Poles $-\omega_0\zeta \pm i\omega_0\sqrt{1-\zeta^2}$ där $|p_1| = |p_2| = \omega_0$.




$\zeta \ll 1 \Rightarrow$ gives resonance peak $\geq |G(i\omega_0)| = \frac{1}{2\zeta}$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# Bode plot
## Example

$$G(s) = \frac{100(s+1)}{s(s^2+6s+100)}$$

- Zeros: $-1$
- Poles:

$$0 \quad \text{and} \quad -3 \pm i\sqrt{91}$$

where $\omega_0 = 10$ and $\zeta = 0.3$.

**[Board: sketch magnitude curve]**


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# Bode plot
## Example

$$G(s) = \frac{100(s+1)}{s(s^2+6s+100)}$$

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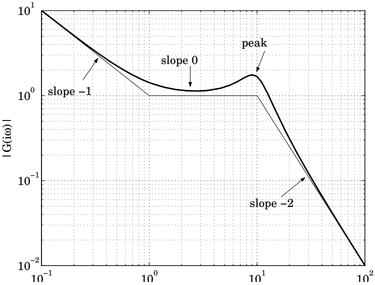


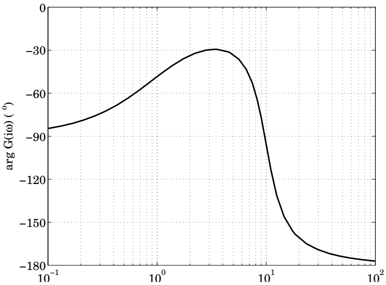




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# Poles/zeros and frequency response
## Bode plot: recipe for phase curve

System on *modified* form:

$$G(s) = K \frac{(1 + \frac{s}{z_1}) \cdots}{s^q (1 + \frac{s}{p_1}) \cdots}$$

*Argument* of frequency response (in radians):

$$\arg\{G(i\omega)\} = -q \cdot \pi + \arctan \frac{\omega}{z_1} + \cdots - \arctan \frac{\omega}{p_1} - \cdots$$



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**Plot phase curve** If $G(s)$ does *not* have any poles/zeros in right half-plane:

- $\underline{\omega \to 0:}$ $\arg G(i\omega) \to 0$ i ing$\frac{1}{s}$ and $G(0) > 0$
- $\underline{\omega \to \infty:}$ each zero or pole contributes with $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$ in phase, respectively.
- Each pole at the origin contributes with $-\frac{\pi}{2}$ in phase $\forall \omega$

Zeros in right half-plane gives negative phase contribution.


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# Controller design in time vs. frequency domain


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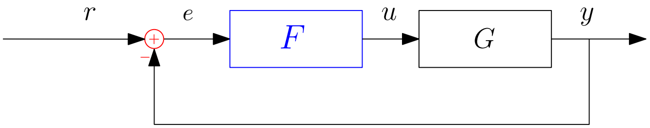
# Design principles for controller

Feedback control system:




- Different performance metrics for $G_c$
- Different design principles of $G_c$ via $F$



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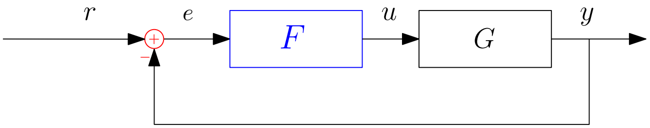
# Design principles for controller

Feedback control system:




- Different performance metrics for $G_c$
- Different design principles of $G_c$ via $F$
- *Ideal controller in the frequency domain:*

$$\boxed{|G_c(i\omega)| \approx 1 \quad \text{and} \quad \arg\{G_c(i\omega)\} \approx 0}$$

for the relevant frequencies $\omega$.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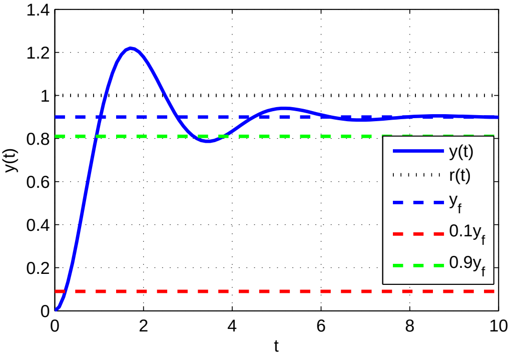
# Controller design in time domain

## Specifications of the time response

$y(t)$ when $r(t)$ is a step $\Leftrightarrow R(s) = \frac{r_0}{s}$:



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# Controller design in time domain

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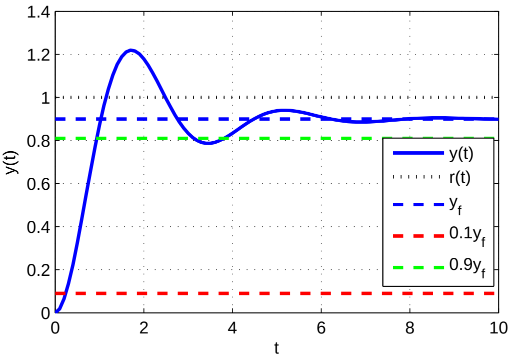


*Performance metrics*

- Quickness: rise time $T_r = t_{90\%} - t_{10\%}$
- Damping: overshoot $M = (y_{\mathsf{max}} - y_f)/y_f$
- Accuracy: static control error $e_f = r_0 - y_f$ (see F3!)



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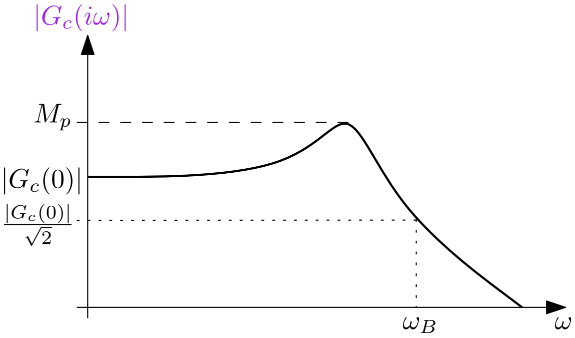
# Controller design in frequency domain
## Specifications of the frequency response

$$|Y(i\omega)| = |G_c(i\omega)||R(i\omega)|$$





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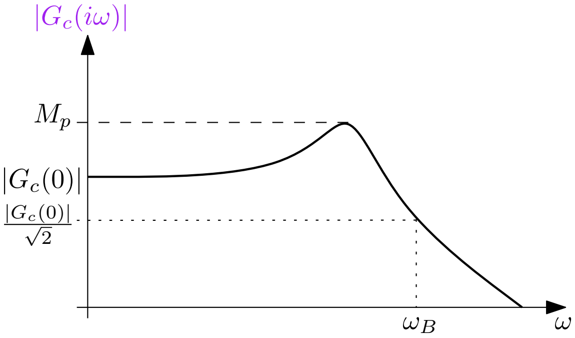


*Performance metrics*

- Quickness: bandwidth $\omega_B$ where $|G_c(i\omega_B)| = |G_c(0)|/\sqrt{2}$
- Damping: resonance peak level $M_p = \max(|G_c(i\omega)|)$
- Accuracy: static gain $G_c(0)$ (see F3!)



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# Controller design via open-loop system


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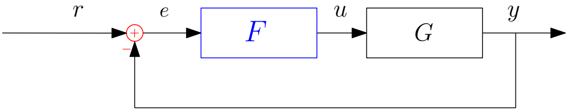
# Design $G_c$ via open-loop system $G_o$

## Frequency response and Nyquist contour

Simple feedback control system:




- Closed-loop: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$
- Open-loop: $G_o(s) = F(s)G(s)$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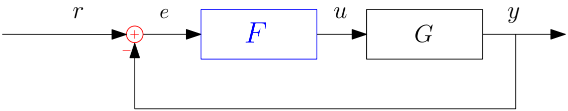


- Closed-loop: $G_c(s) = \frac{G_o(s)}{1+G_o(s)}$
- Open-loop: $G_o(s) = F(s)G(s)$
- $G_c(s)$ stable $\iff 1 + G_o(s)$ no roots in right half-plane.

**Nyquist contour**

Plot $G_o(i\omega)$, as a function of $0 \le \omega < \infty$.


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# Nyquist contour

## Example

DC-motor $G(s) = \frac{1}{s(s+1)}$ with controller $F(s) = \frac{K}{s+2}$



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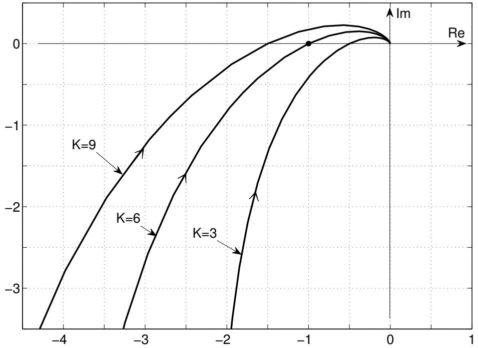
# Nyquist contour

## Example

DC-motor $G(s) = \frac{1}{s(s+1)}$ with controller $F(s) = \frac{K}{s+2}$

Nyquist contour for $G_o(i\omega) = G(i\omega)F(i\omega)$:




Then $G_c$: $K = 3$ (stable), $K = 9$ (unstable), and $K = 6$ (marginally stable).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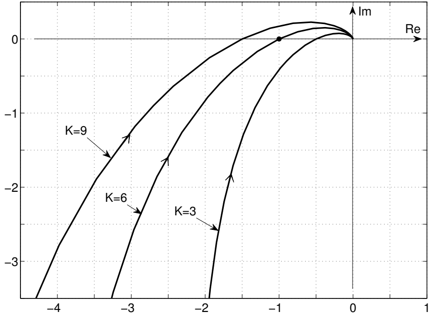
# Nyquist criterion

## Design $G_c$ via $G_o$




**(Result 3.3) Basic Nyquist criterion:**

If $G_o(s)$ has no poles in right half-plane:

$G_c(s)$ stable $\Leftrightarrow$ *Nyquist contour* $G_o(i\omega)$ does not encircle $-1$



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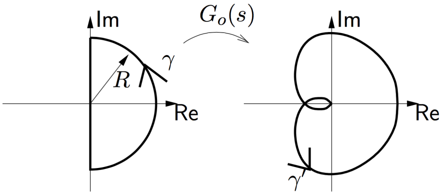
# Nyquist criterion

*In general:* Let $s$ form semi-circle $\gamma$, with radius $R \to \infty$. Then Nyquist contour $G_o(s)$ is $\gamma'$.





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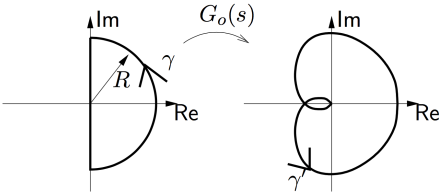


(Result 3.3) General Nyquist criterion:

$$\#\text{poles}\{G_c(s)\} \text{ in right half-plane}$$
$$=$$
$$\#\text{poles}\{G_o(s)\} \text{ in right half-plane}$$
$$+ \text{ Number of positive circles of } \gamma' \text{ around } -1$$


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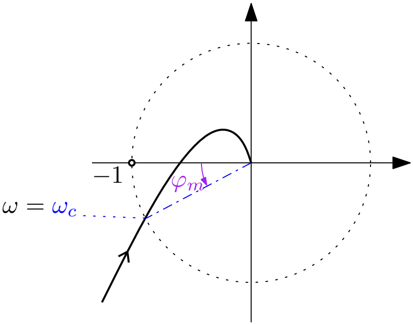
# Controller design via Nyquist/Bode plot

## Specification via $G_o(s)$

$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \frac{|G_o(i\omega)|}{\underbrace{|G_o(i\omega) - (-1)|}_{\text{distance to } -1}}$$

Nyquist contour $G_o(i\omega)$



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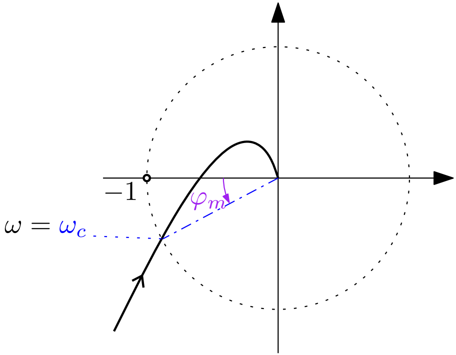
# Controller design via Nyquist/Bode plot

## Specification via $G_o(s)$

Nyquist contour $G_o(i\omega)$:




### Crossover frequency and phase margin

Find $\omega = \omega_c$ where $|G_o(i\omega_c)| = 1$ and then define
$\varphi_m = \arg\{G_o(i\omega_c)\} + 180°$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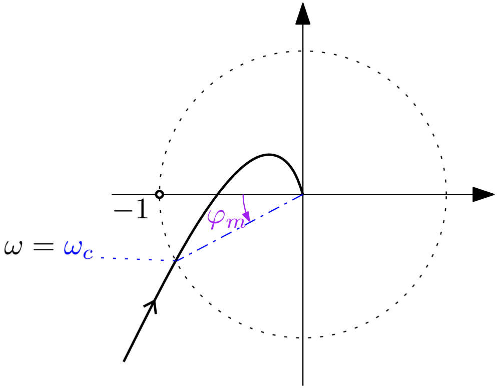
# Controller design via Nyquist/Bode plot

## Specification via $G_o(s)$




Open-loop *design* $G_o(i\omega)$:

- Crossover frequency $\omega_c$ ⇒ bandwidth $\omega_B$ of $G_c$ (quickness)
- Phase margin $\varphi_m$ ⇒ resonance peak $M_p$ of $G_c$ (damping)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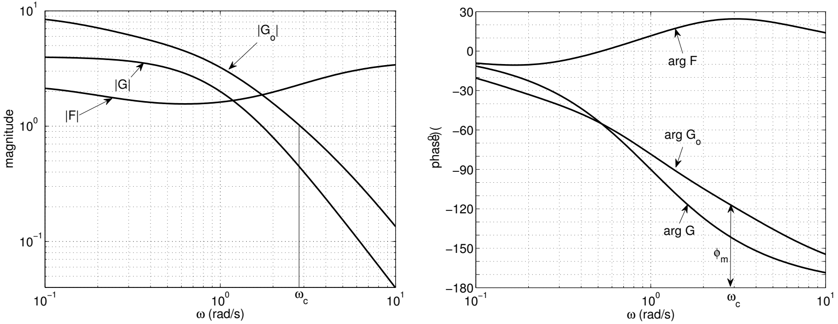
# Controller design via Nyquist/Bode plot

## Specification via $G_o(s)$

Open-loop frequency characteristics that we can shape:

- Crossover frequency $\omega_c$
- Phase margin $\varphi_m$




Note $\log_{10}|G_o| = \log_{10}|G| + \log_{10}|F|$ and
$\arg\{G_o\} = \arg\{G\} + \arg\{F\}$



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# Design in the frequency domain

## Compensate P-controller in frequency domain

Control structure starting with P-control:

$$F(s) = \underbrace{K}_{\text{P-control}} F_{\text{lead}}(s) F_{\text{lag}}(s)$$


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# Design in the frequency domain

## Compensate P-controller in frequency domain

Control structure starting with P-control:

$$F(s) = \underbrace{K}_{\text{P-control}} F_{\text{lead}}(s) F_{\text{lag}}(s)$$

where

- $F_{\text{lead}}(s)$ adjusts $\varphi_m$ (damping) and $\omega_c$ (quickness)
- $F_{\text{lag}}(s)$ adjusts $G_o(0)$ (accuracy)

*Recall that*

$$|G_o(i\omega)| = |G(i\omega)|K|F_{\text{lead}}(i\omega)||\{F_{\text{lag}}(i\omega)|$$
$$\arg\{G_o(i\omega)\} = \arg\{G(i\omega)\} + \arg\{F_{\text{lead}}(i\omega)\} + \arg\{F_{\text{lag}}(i\omega)\},$$

shape the Bode plot/Nyquist contour of $G_o$



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# Design in the frequency domain

## Compensate P-controller in frequency domain

Control structure starting with P-control:

$$F(s) = \underbrace{K}_{\text{P-control}} F_{\text{lead}}(s) F_{\text{lag}}(s)$$

- Lead filter

$$F_{\text{lead}}(s) = \frac{\tau_D s + 1}{\beta \tau_D s + 1}, \quad 0 \leq \beta < 1, \quad \tau_D > 0$$

increases phase margin $\varphi_m$ and crossover $\omega_c$.



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- Lag filter

$$F_{\text{lag}}(s) = \frac{\tau_I s + 1}{\tau_I s + \gamma}, \quad 0 \leq \gamma < 1, \quad \tau_I > 0$$

  increases static gain $G_c(0)$.
- See ch. 5.4 for tuning principles


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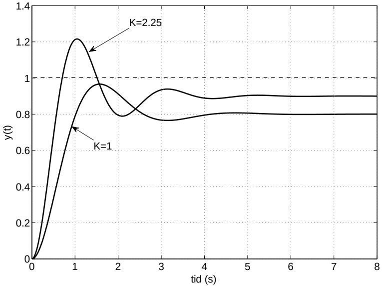
# Lead-lag design using Bode plot

## Example of lead-lag controller

System $Y(s) = \frac{4}{(s+1)^2}U(s)$ with controller
$U(s) = F(s)(R(s) - Y(s))$ where $F(s) = K$.




*Performance specifications* when $r(t)$ is a step:

- Accuracy, static error $e_f \leq 0.1$,
- Quickness, rise time $T_r \leq 0.5$,
- Damping, overshoot $M \leq 20\%$.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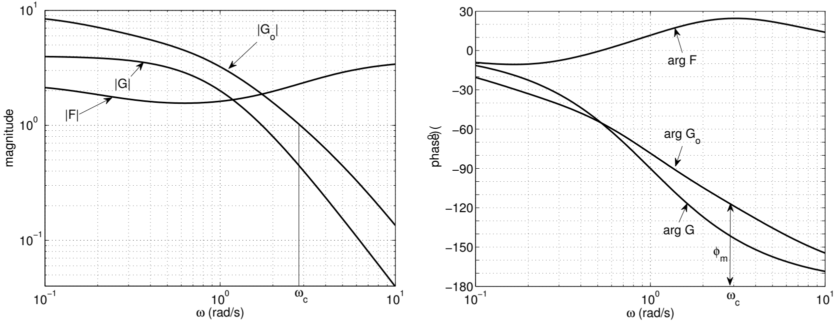
# Lead-lag design using Bode plot

## Example of lead-lag controller

Bode plots for

- system $G(s)$,
- open-loop $G_o(s) = F(s)G(s)$,
- build lead-lag controller $F(s)$ that compensates $P$-part



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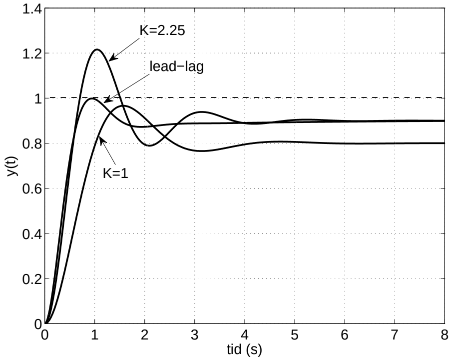
# Lead-lag design using Bode plot

## Example of lead-lag controller

Step response $y(t)$ with lead-lag controller

$$F(s) = K F_{\text{lead}}(s) F_{\text{lag}}(s).$$




Performance specifications are met with lead-lag controller:

- $T_r = 0.45$ seconds, $M = 8\%$, $e_f < 0.1$.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# Minimum phase systems



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# Minimum phase systems

Several different systems $G(s)$ may have identical magnitude curve $|G(i\omega)|$. Only phase curve $\arg\{G(i\omega)\}$ will differ.

**Definition:**

Among all systems $G(s)$ with the *same magnitude curve*, the system with the *least* negative phase shift is a minimum phase system.


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**Result 5.1**

A system is minimum phase $\Leftrightarrow$ it has no poles nor zeros in the right half-plane, and contains no time delays.


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**Result 5.1**

A system is minimum phase $\Leftrightarrow$ it has no poles nor zeros in the right half-plane, and contains no time delays.

Cf. bicycle as non-minimum phase!


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# Non-minimum phase systems

## Example

Exemple of open-loop systems

- $G_o(s) = \frac{s+1}{s(s^2+2s+2)}$: $p_i = 0, -1 \pm i$. $z_i = -1$
- $G'_o(s) = \frac{-s+1}{s+1} \cdot G_o(s)$: $p_i = 0, -1 \pm i$. $z_i = +1$
- $G''_o(s) = e^{-2s} G_o(s)$: as above but time-delay 2 sec.



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# Non-minimum phase systems
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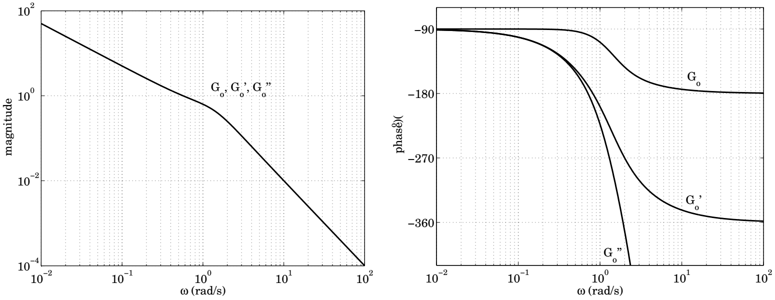
Bode plots for $G_o$, $G'_o$ & $G''_o$:



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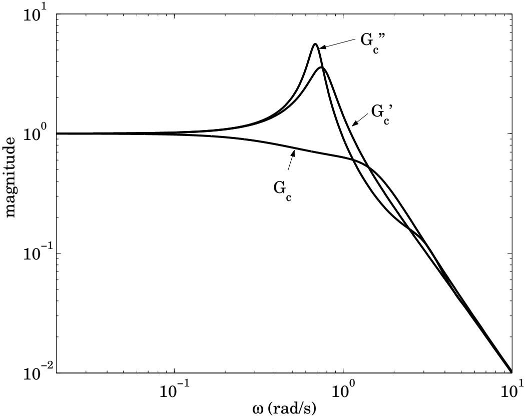
# Non-minimum phase systems

## Examples

Magnitude curves for corresponding closed-loop systems, $G_c$, $G'_c$ & $G''_c$:




Non-minimum phase systems are hard to control!



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# Summary and recap

- Frequency response and Bode plots
- Performance metrics in the frequency domain
  - Bandwidth
  - Resonance peak
  - Static gain
- Open-loop design and Nyquist contour
- Minimum phase systems



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