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Intro. Computer Control Systems: F7

State-space descriptions of systems

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F6: Quiz!

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F6: Quiz!

1) The bandwidth ω_B of the closed-loop system $G_c(s)$ affects its
   a quickness $\uparrow$
   b damping $\uparrow$
   c stability $\downarrow$

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F6: Quiz!

1) The bandwidth ω_B of the closed-loop system $G_c(s)$ affects its
   - a quickness $\uparrow$
   - b damping $\uparrow$
   - c stability $\downarrow$

2) A non-minimum phase system is
   - a always unstable $\uparrow$
   - b may have a zero in the right half-plane $\uparrow$
   - c easy to control $\downarrow$

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Systems in state-space description

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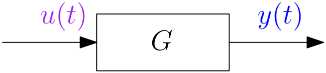
Linear time-invariant systems



Different mathematical forms of *same model*:

- ODE:
$$\frac{d^n}{dt^n}y + \cdots + a_{n-1}\frac{d}{dt}y + a_n y = b_0\frac{d^m}{dt^m}u + \cdots + b_{m-1}\frac{d}{dt}u + b_m u$$

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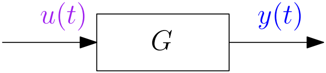
Linear time-invariant systems



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- Transfer function:

$$Y(s) = G(s)U(s) \quad \text{ignoring initial conditions}$$

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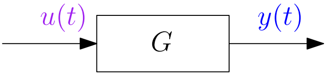
Linear time-invariant systems



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- Transfer function:

$$Y(s) = G(s)U(s) \quad \text{ignoring initial conditions}$$

- State-space description:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

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System states

System states at time t $=$ summary of its history to predict effect of input $u(t^*)$ where $t^* > t$.

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- Transfer function yields

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_{\tau=0}^{t} g(\tau) \underbrace{u(t-\tau)}_{\text{past input}} d\tau$$

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- Alternative description using n state variables

$$y(t) = \sum_{i=1}^{n} c_i \underbrace{x_i(t)}_{\text{state variables}} + Du(t) = Cx(t) + Du(t)$$

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where

$$C = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \quad \text{and} \quad x(t) \triangleq \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

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Note: matrix multiplication and eigenvalues necessary!

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Building intuition

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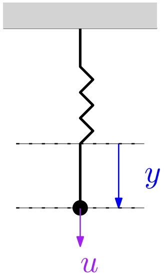
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$



Figur : Force $u(t)$ and position $y(t)$.

Standard form of linear ODE model:

$$\frac{d^2}{dt^2}y + \left(\frac{K}{m}\right)y = \left(\frac{1}{m}\right)u$$

[Board: derive state-space description]

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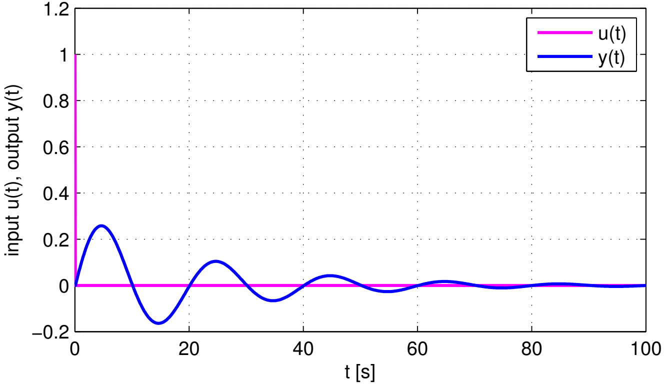
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$

Input $u(t)$ impulse



Output $y(t) = Cx(t)$

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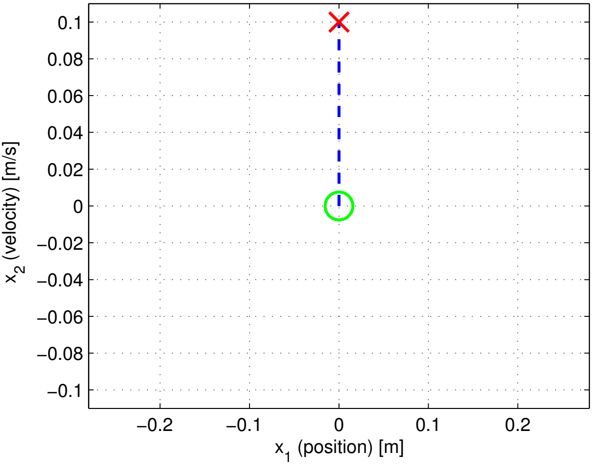
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$



$x(t)$ at $t = 0^+$

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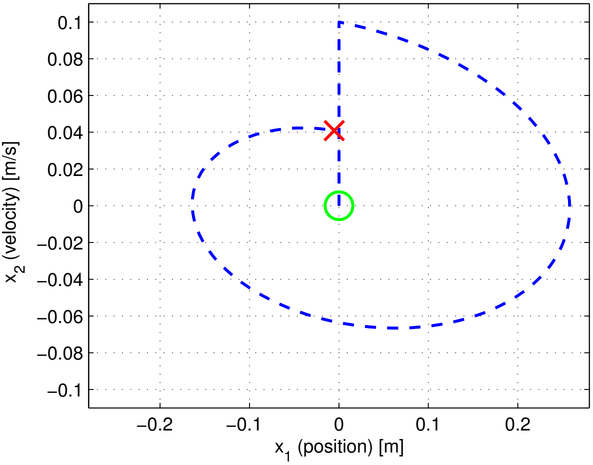
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$



$x(t)$ at $t = 20$

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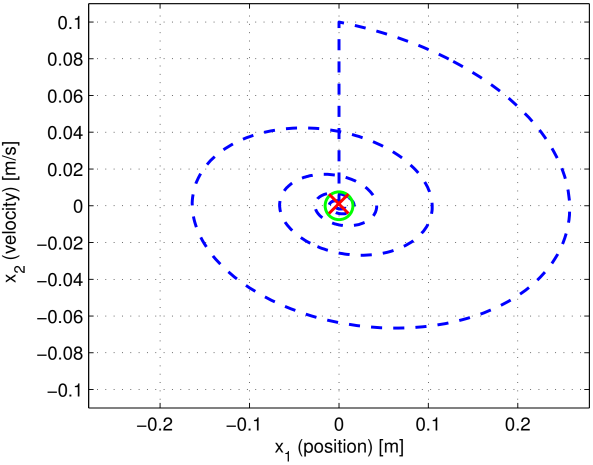
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$



$x(t)$ at $t = 100$

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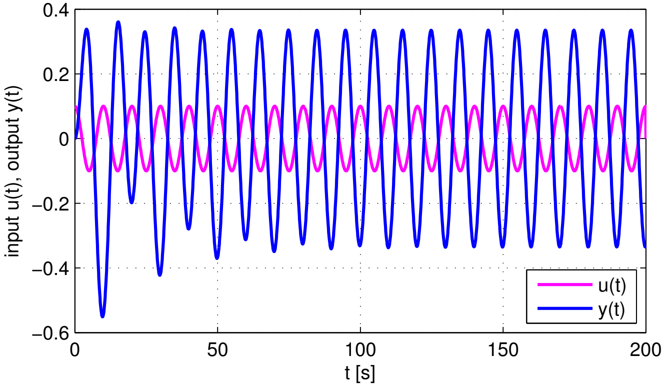
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$

Input $u(t)$ sine



Output $y(t) = Cx(t)$

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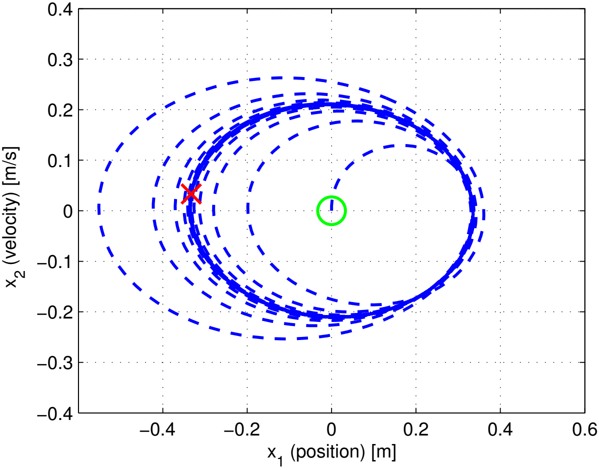
Build intuition from simple systems

Example: State vector in space $\mathbb{R}^2$



$x(t)$ at $t = 200$

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Relation between system descriptions

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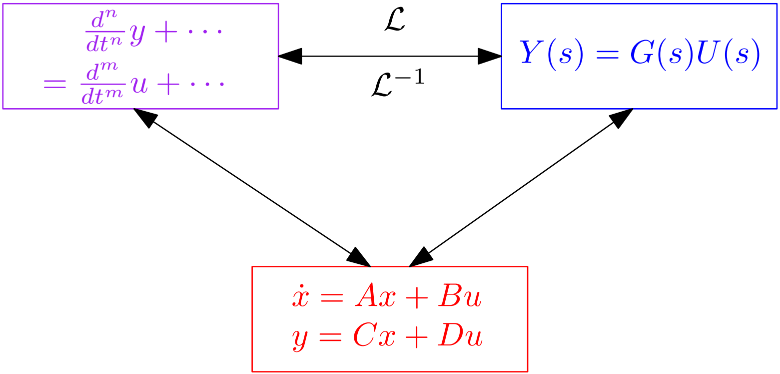
Linear time-invariant system models

Relations between mathematical descriptions

Descriptions with different strengths



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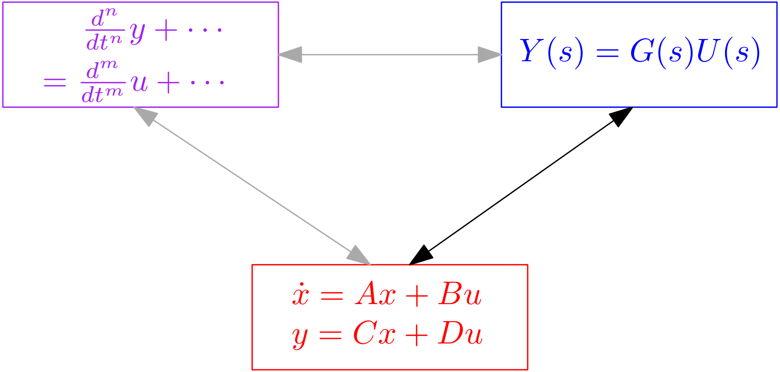
Linear time-invariant system models

Relations between mathematical descriptions

Translate from one description to the next



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Relations between descriptions

State-space form → transfer function

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$Y(s) = G(s)U(s)$$

[Board: Laplace transform and solve $G(s)$]

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Relations between descriptions

State-space form $\rightarrow$ transfer function

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \quad \longrightarrow \quad Y(s) = G(s)U(s)$$

Transfer function obtained by

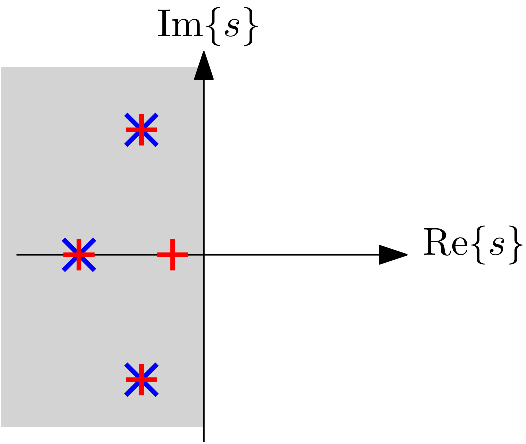
$$G(s) = \underbrace{C}_{1\times n}\underbrace{(sI - A)^{-1}}_{n\times n}\underbrace{B}_{n\times 1} + \underbrace{D}_{1\times 1} = \frac{b(s)}{a(s)}$$

Important property:

- System matrix A:s eigenvalues $\{\lambda_i\}$ given by solution to $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots a_n = 0$
- $a(s) = \det(sI - A)$ is a polynomial of order n

Relations between descriptions

State-space form → transfer function



$G(s)$:s poles $p_i \subseteq A$:s eigenvalues λ_j

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Relations between descriptions

Transfer function $\rightarrow$ State-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\longleftarrow$

$$Y(s) = G(s)U(s)$$

Choice of state variables and system matrices not unique!

[Board: alternative states $z = Tx$]

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Relations between descriptions

Transfer function $\to$ State-space form

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

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Relations between descriptions

Transfer function → State-space form

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

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Relations between descriptions

Transfer function → State-space form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \quad \longleftarrow \quad Y(s) = G(s)U(s)$$

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

we have general state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Note: when order of numerator $< n$ we obtain $D = 0$

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Solving the state-space equation

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Solution to state-space equation
First-order system

Output $y(t) = cx(t) + du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0.$$

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Solution to state-space equation

First-order system

Output $y(t) = cx(t) + du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0.$$

Via Laplace transform of both sides one obtains

$$X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$

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Via Laplace transform of both sides one obtains

$$X(s) = \frac{1}{s-a}x_0 + \frac{b}{s-a}U(s)$$

and inverse transform yields solution

$$\boxed{x(t) = e^{at}x_0 + \int_0^t \underbrace{e^{a\tau}b}_{h(\tau)} u(t-\tau)d\tau}$$

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Solution to state-space equation
Matrix exponential

- Exponential e^{at} is a function which fulfills

$$\dot{f}(t) = af(t), \quad f(0) = 1$$

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Solution to state-space equation

Matrix exponential

- Exponential e^{at} is a function which fulfills

$$\dot{f}(t) = af(t), \quad f(0) = 1$$

- Via Laplace transform

$$F(s) = \frac{1}{s-a} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

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Matrix exponential

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- Matrix exponential e^{At} is a matrix function which fulfills

$$\dot{F}(t) = AF(t), \quad F(0) = I$$

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Solution to state-space equation
Matrix exponential

- Exponential e^{at} is a function which fulfills

$$\dot{f}(t) = af(t), \quad f(0) = 1$$

- Via Laplace transform

$$F(s) = \frac{1}{s-a} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad e^{at} = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

- Matrix exponential e^{At} is a matrix function which fulfills

$$\dot{F}(t) = AF(t), \quad F(0) = I$$

- Via Laplace transform

$$F(s) = (sI - A)^{-1} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad \boxed{e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

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Solution to state-space equation

General

Output $y(t) = Cx(t) + Du(t)$ where $x(t)$ given by solution to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

[Board: Laplace + inverse transform]

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Solution to state-space equation

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[Board: Laplace + inverse transform]

Via Laplace transform of both sides

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}BU(s)$$

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Solution to state-space equation

General

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[Board: Laplace + inverse transform]

Via Laplace transform of both sides

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and inverse transform gives solution

$$\boxed{x(t) = e^{At}x_0 + \int_0^t \underbrace{e^{A\tau}B}_{H(\tau)} u(t-\tau)d\tau}$$

Note: matrix exponential e^{At}

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Stable states and stable systems

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Stability

The state evolve according to:

$$x(t) = e^{At}x_0 + \int_0^t e^{A\tau} B u(t-\tau) d\tau$$

Asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \quad \text{when } u(t) \equiv 0$$

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Asymptotic stability of states

A:s eigenvalues are strictly in left half-plane $\Leftrightarrow$ system is asymptotically stable

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Stability

The state evolve according to:

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Asymptotically stable if

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Asymptotic stability of states

A:s eigenvalues are strictly in left half-plane $\Leftrightarrow$ system is asymptotically stable

Input-output stability of system

A:s eigenvalues are strictly in left half-plane $\Rightarrow$ system is input-output stable

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Summary and recap

- State-space description using vectors and matrices
- System matrices and transfer functions
- Solution to state-space equation
- Stability concepts

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