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Intro. Computer Control Systems: F8

Properties of state-space descriptions and feedback

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F7: Quiz!

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1) The state-space description of a system is
   - a not unique $\uparrow$
   - b unique $\uparrow$
   - c stable $\downarrow$

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F7: Quiz!

1) The state-space description of a system is
   a not unique $\uparrow$
   b unique $\uparrow$
   c stable $\downarrow$

2) The eigenvalues of the system matrix A reveals something about
   a poles $\uparrow$
   b zeros $\uparrow$
   c the closed-loop system $\downarrow$

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F7: Quiz!

1) The state-space description of a system is
   - a not unique $\uparrow$
   - b unique $\uparrow$
   - c stable $\downarrow$

2) The eigenvalues of the system matrix A reveals something about
   - a poles $\uparrow$
   - b zeros $\uparrow$
   - c the closed-loop system $\downarrow$

3) Solution to $\dot{x} = Ax + Bu$ with initial condition x_0 is obtained using
   - a a linear system of equations $\uparrow$
   - b the matrix exponential $\uparrow$
   - c the Nyquist contour $\downarrow$

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Nonlinear time-invariant systems

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Nonlinear systems and states

Most systems are nonlinear!

Nonlinear differential equations:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

Linearize around *operating point* x_0, u_0. Typically use a stationary point: $\dot{x} = f(x_0, u_0) = 0$

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Nonlinear systems and states

Nonlinear differential equations:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

Taylor series expansion around *stationary point* x_0, u_0 with $y_0 = h(x_0, u_0)$ results in linear deviation model:

$$\boxed{\begin{aligned} \dot{\Delta} x &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x + D\Delta u \end{aligned}}$$

- Linear state-space description of the deviations around the operating point of system.
- Matrices A, B, C and D given by derivatives of $f(x, u)$ and $h(x, u)$ with respect to x and u. See ch. 8.4 G&L.

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Feedback control using states

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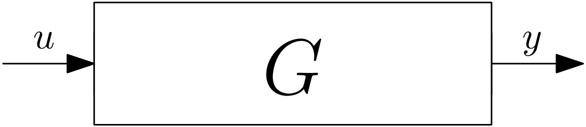
State-feedback control

State space description of linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \Rightarrow \quad Y(s) = G(s)U(s)$$



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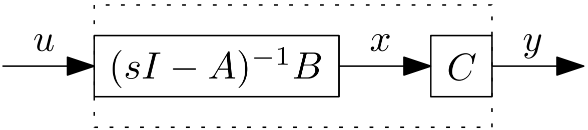
State-feedback control

State space description of linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \Rightarrow \quad G(s) = C(sI - A)^{-1}B$$



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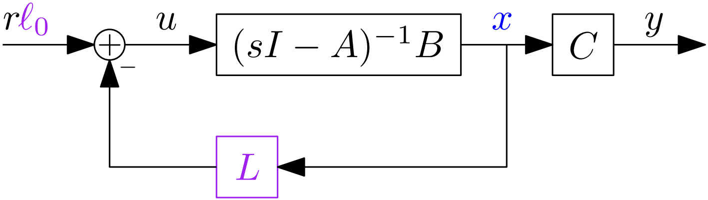
State-feedback control

Idea: Feedback control using states

$$\boxed{u = -Lx + \ell_0 r,}$$

where L and ℓ_0 are design parameters.



$$\dot{x} = Ax + B\underbrace{(-Lx + l_0 r)}_{=u}$$

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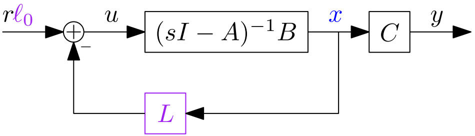
State-feedback control



Closed-loop system from r to y comes:

$$\dot{x} = Ax + B(-Lx + l_0 r) = (A - BL)x + Bl_0 r$$
$$y = Cx$$

Is it possible to

- control the system to *all* states x^* in $\mathbb{R}^n$?
- design the *closed-loop system's* poles?
- (estimate the state $x(t)$?)

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Controllability

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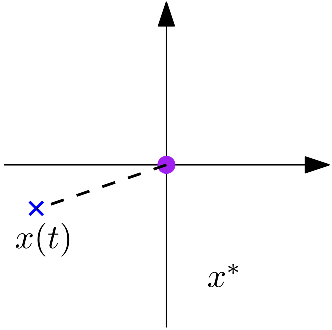
Controllability

A sought state x^* is controllable if some input $u(t)$ can move the system from $x(0) = 0$ to $x(T) = x^*$



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Controllability

For $x_0 = 0$, we can compute the state at $t = T$

$$x(T) = e^{At}x_0 + \int_0^T e^{A\tau} B u(T - \tau) d\tau$$

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Controllability

Med $x_0 = 0$ är tillståndet vid $t = T$

$$
\begin{aligned}
x(T) &= \int_0^T e^{A\tau} B u(T-\tau) d\tau \\
&= \lceil \text{via Cayley-Hamiltons theorem} \rceil \\
&= B\gamma_0 + AB\gamma_1 + \cdots + A^{n-1}B\gamma_{n-1}
\end{aligned}
$$

Therefore:

- $x(T)$ is a linear combination of $B, AB, \ldots, A^{n-1}B$.
- A state x^* is controllable if it can be expressed as such a linear combination, i.e., if x^* is in the column space of

$$
\boxed{\mathcal{S} \triangleq [B \; AB \; \cdots \; A^{n-1}B]}
$$

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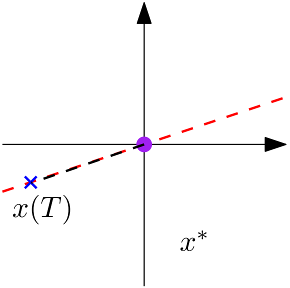
Controllability



Figur : Example column space of $\mathcal{S}$ and non-controllable state x^*.

Controllable system

All states x^* are controllable $\Leftrightarrow$ $\mathcal{S}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{S}) = n$ or $\det(\mathcal{S}) \neq 0$

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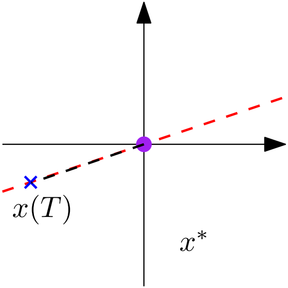
Controllability



Figur : Example column space of $\mathcal{S}$ and non-controllable state x^*.

Controllable canonical form

System is controllable $\Leftrightarrow$ It can be written on controllable canonical form

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Observability

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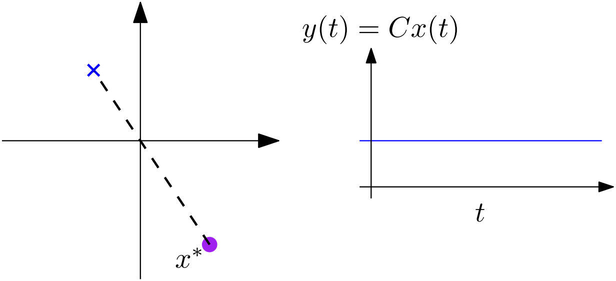
Observability

Assume $u(t) \equiv 0$. A state $x^* \neq 0$ is unobservable if the output $y(t) \equiv 0$ when system starts at $x(0) = x^*$.



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Observability

When $u(t) \equiv 0$ we obtain

$$\begin{aligned} y(t) &= Cx(t) \\ &= Ce^{At}x^* + 0 \end{aligned}$$

When $y(t) \equiv 0$ we do not observe any changes in the output:

$$\left.\frac{d^k}{dt^k} y(t)\right|_{t=0} = CA^k x^* = 0.$$

That is,

$$Cx^* = 0, \quad CAx^* = 0, \quad \ldots, \quad CA^{n-1}x^* = 0$$

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Observability

When $u(t) \equiv 0$ and $y(t) \equiv 0$ we observe no changes:

$$Cx^* = 0, \quad CAx^* = 0, \quad \dots, \quad CA^{n-1}x^* = 0$$

or

$$\mathcal{O}x^* = 0$$

where

$$\boxed{\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}$$

Therefore:

- A state $x^* \neq 0$ is unobservable if it belongs to the null space of $\mathcal{O}$.

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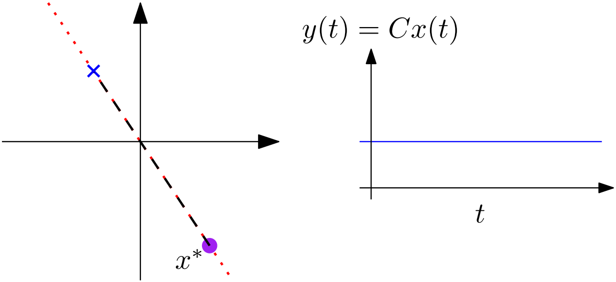
Observability



Figur : Example null space of $\mathcal{O}$ and unobservable state x^*.

Observable system

All states x^* are observable $\Leftrightarrow$ $\mathcal{O}$:s columns are linearly independent

Note: $\text{rank}(\mathcal{O}) = n$ or $\det(\mathcal{O}) \neq 0$

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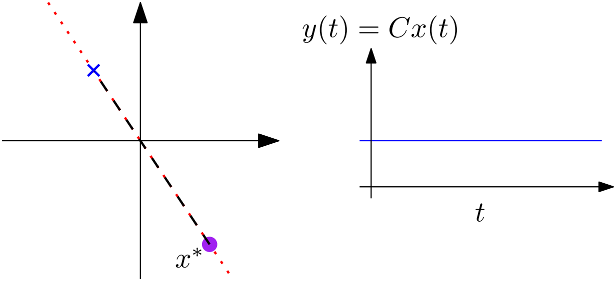
Observability



Figur : Example null space of $\mathcal{O}$ and unobservable state x^*.

Observable canonical form

System is observable $\Leftrightarrow$ It can be written on observable canonical form

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Build intuition

Build intuition from simple systems

Example: controllable system

System on controllable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

Transfer function:

$$G(s) = C(sI - A)^{-1}B = \frac{s+1}{s^2+2s+1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$$

[Board: investigate observability using $\mathcal{O}$]

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Build intuition from simple systems

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[Board: investigate observability using $\mathcal{O}$]

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \det \mathcal{O} = 0 \quad \Leftrightarrow \quad \text{unonbservable}$$

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Build intuition from simple systems

Example: observable system

System on observable canonical form:

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
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Transfer function:

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[Board: investigate controllability using $\mathcal{S}$]

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Build intuition from simple systems

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[Board: investigate controllability using $\mathcal{S}$]

$$\mathcal{S} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \det \mathcal{S} = 0 \quad \Leftrightarrow$$ non-controllable

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Build intuition from simple systems

Exemple: controllable and observable system

Systems in previous examples have the same transfer function

$$G(s) = \frac{1}{s+1}.$$

Can also be written in state-space form

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t), \\ y(t) &= x(t). \end{aligned}$$

where $x(t)$ is a scalar.

[Board: investigate $\mathcal{S}$ and $\mathcal{O}$]

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Build intuition from simple systems

Exemple: controllable and observable system

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where $x(t)$ is a scalar.

[Board: investigate $\mathcal{S}$ and $\mathcal{O}$]

$$\begin{matrix} \mathcal{S} = 1 \\ \mathcal{O} = 1 \end{matrix} \quad \Rightarrow \begin{matrix} \det \mathcal{S} = 1 \\ \det \mathcal{O} = 1 \end{matrix} \quad \Leftrightarrow \quad \text{controllable and observable} \qquad (1)$$

Note: we eliminated "invisible states"

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Minimal realization

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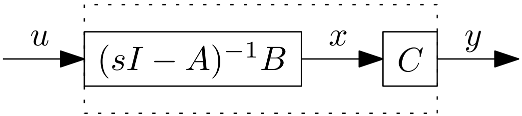
Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$



Definition 8.2 G&L

State-space form of $G(s)$ is a minimal realization if vector x has the smallest possible dimension.

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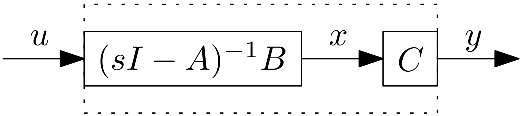
Minimal realization

System with transfer function $G(s)$ and state-space form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$



Definition 8.2 G&L

State-space form of $G(s)$ is a minimal realization if vector x has the smallest possible dimension.

Result 8.11(+8.12) G&L

A state-space form is minimal realization $\Leftrightarrow$ controllable and observable $\Leftrightarrow$ A:s eigenvalues $=$ $G(s)$:s poles

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Design of state-feedback control

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State-feedback control

State-space model with controller $u = -Lx + \ell_0 r$ where

$$L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}$$

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State-feedback control

State-space model with controller $u = -Lx + \ell_0 r$ where

$$L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}$$

Closed-loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + B\ell_0 r \\ y &= Cx \end{aligned}$$

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State-feedback control

State-space model with controller $u = -Lx + \ell_0 r$ where

$$L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}$$

Closed-loop system as a transfer function

Output is $Y(s) = G_c(s)R(s)$, where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0$$

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State-feedback control

State-space model with controller $u = -Lx + \ell_0 r$ where

$$L = \begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_n \end{bmatrix}$$

System matrix of closed-loop system:

$$\boxed{(A - BL)}$$

Eigenvalues/poles given by polynomial equation

$$\boxed{\det(sI - A + BL) = 0}$$

which we can *design* via L!

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State-feedback control

Design of the gain ℓ_0

- $Y(s) = G_c(s)R(s)$ where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- It is *desirable* to have at least $G_c(0) = 1$

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State-feedback control

Design of the gain ℓ_0

- $Y(s) = G_c(s)R(s)$ where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- It is *desirable* to have at least $G_c(0) = 1$
- $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so

$$\boxed{\ell_0 = \frac{1}{C(-A + BL)^{-1}B}}$$

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State-feedback control

Design of the gain ℓ_0

- $Y(s) = G_c(s)R(s)$ where

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

- It is *desirable* to have at least $G_c(0) = 1$
- $G_c(0) = C(-A + BL)^{-1}B\ell_0 = 1$ and so

$$\boxed{\ell_0 = \frac{1}{C(-A + BL)^{-1}B}}$$

- More generally, replace $\ell_0 r$ with $F_r(s)R(s)$
- How to design L?

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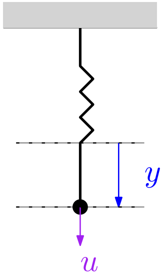
Build intuition from simple systems

Exemple: state-vector in $\mathbb{R}^2$



Figur : Force $u(t)$ and position $y(t)$.

State-space form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

[Board: design L so that closed-loop system has poles -2 and -3]

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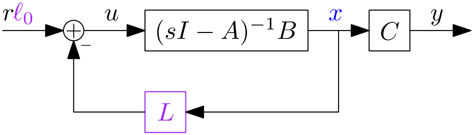
Pole placement

State-feedback control



Result 9.1

State-space form is controllable $\Leftrightarrow$ L can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system

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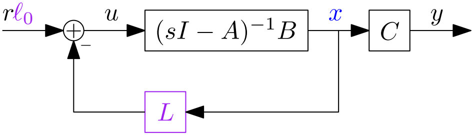
Pole placement

State-feedback control



Result 9.1

State-space form is controllable $\Leftrightarrow$ L can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system

- L solved by $\det(sI - A + BL) = 0$ with desired roots
- L very simple to solve for system on controllable canonical form

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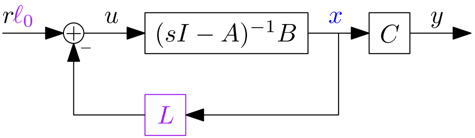
Pole placement

State-feedback control



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- L solved by $\det(sI - A + BL) = 0$ with desired roots
- L very simple to solve for system on controllable canonical form

What to do when we can't measure x directly?

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Summary and recap

- Linearization of nonlinear system models
- Properties:
  - Controllable
  - Observable
  - Minimal realization
- State-feedback control
- Pole placement for the closed-loop system

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