Intro. Computer Control Systems: F9

State-feedback control and observers

Dave Zachariah

Dept. Information Technology, Div. Systems and Control

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

F8: Quiz!

dave.zachariah@it.uu.se

F8: Quiz!

1) For an observable system
   - a the effect of all $x(t)$ can be observed in $y(t)$ $\uparrow$
   - b we have $\det \mathcal{O} = 0$ $\uparrow$
   - c we have stability $\downarrow$

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

F8: Quiz!

1) For an observable system
   - a the effect of all $x(t)$ can be observed in $y(t)$ $\uparrow$
   - b we have $\det \mathcal{O} = 0$ $\uparrow$
   - c we have stability $\downarrow$

2) If a state-space form of $G(s)$ is a minimal realization,
   - a A:s eigenvalues $<$ $G(s)$:s poles $\uparrow$
   - b A:s eigenvalues $=$ $G(s)$:s poles $\uparrow$
   - c there exist more compact state-space forms $\downarrow$

dave.zachariah@it.uu.se

F8: Quiz!

1) For an observable system
   - a the effect of all $x(t)$ can be observed in $y(t)$ $\uparrow$
   - b we have $\det \mathcal{O} = 0$ $\uparrow$
   - c we have stability $\downarrow$

2) If a state-space form of $G(s)$ is a minimal realization,
   - a A:s eigenvalues $<$ $G(s)$:s poles $\uparrow$
   - b A:s eigenvalues $=$ $G(s)$:s poles $\uparrow$
   - c there exist more compact state-space forms $\downarrow$

3) For a controllable system with state-feedback control
   - a no information about the system is required $\uparrow$
   - b the poles of the closed-loop system can be designed arbitrarily $\uparrow$
   - c the closed-loop system is stable $\downarrow$

dave.zachariah@it.uu.se

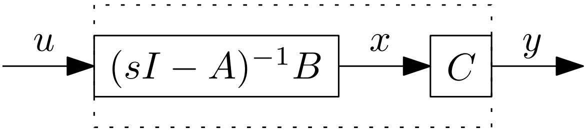
State-feedback control

dave.zachariah@it.uu.se

State-feedback control

State-space form of linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \Rightarrow \quad G(s) = C(sI - A)^{-1}B$$



dave.zachariah@it.uu.se

State-feedback control

Controler using state feedback

$$u = -Lx + \ell_0 r$$

gives closed-loop system

$$\begin{aligned}\dot{x} &= (A - BL)x + B\ell_0 r \\ y &= Cx\end{aligned}$$

where r is the reference signal.

dave.zachariah@it.uu.se

State-feedback control

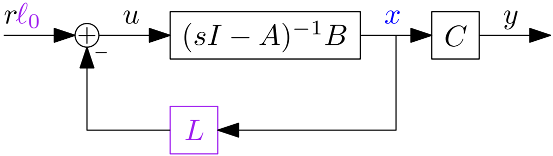
Controler using state feedback

$$u = -Lx + \ell_0 r$$

gives closed-loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + B\ell_0 r \\ y &= Cx \end{aligned}$$

where r is the reference signal.



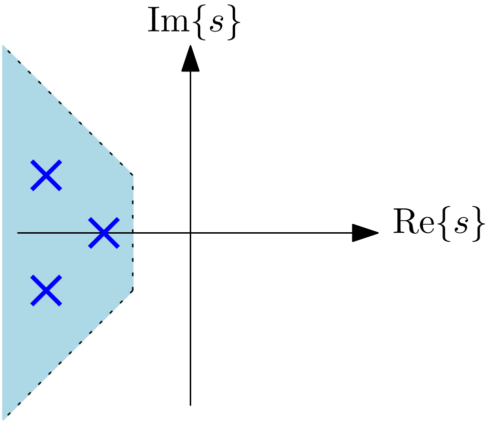
$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0$$

dave.zachariah@it.uu.se

Pole placement

Rules of thumb for designing L

Eigenvalues/poles given by $\det(sI - A + BL) = 0$, which we can *design*

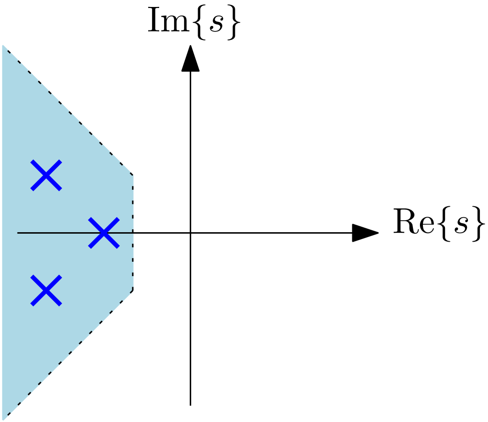


dave.zachariah@it.uu.se

Pole placement

Rules of thumb for designing L

Eigenvalues/poles given by $\det(sI - A + BL) = 0$, which we can *design*



Distance to the origin: Quick system but also sensitive to disturbances

dave.zachariah@it.uu.se

Estimating the states via simulation

dave.zachariah@it.uu.se

Estimating the states

Via simulation

- Controller

$$u = -Lx + \ell_0 r$$

requires states x which are often unknown.

dave.zachariah@it.uu.se

Estimating the states

Via simulation

- Controller

$$u = -Lx + \ell_0 r$$

requires states x which are often unknown.

- In practice, feedback using

$$\boxed{u = -L\hat{x} + \ell_0 r}$$

where $\hat{x}$ is an estimate of x.

dave.zachariah@it.uu.se

Estimating the states

Via simulation

- Controller

$$u = -Lx + \ell_0 r$$

requires states x which are often unknown.

- In practice, feedback using

$$\boxed{u = -L\hat{x} + \ell_0 r}$$

where $\hat{x}$ is an estimate of x.

- *Naive idea:* Estimate x by *simulating* the states

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \hat{x}(0) = \hat{x}_0$$

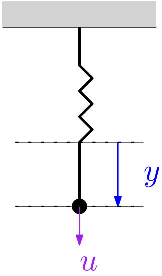
where $\hat{x}_0$ is an initial guess.

dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Ex.: Damper



State-space form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t), \quad x(0) = x_0$$

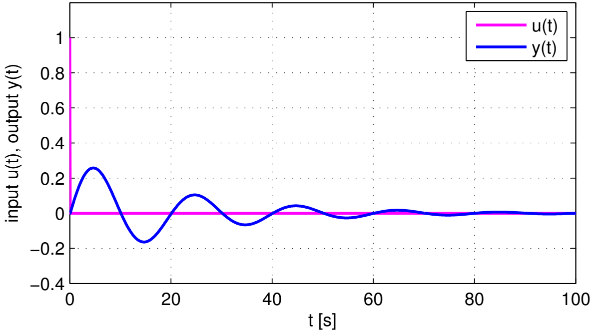
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Example using impulse u



System with unknown initial state x_0

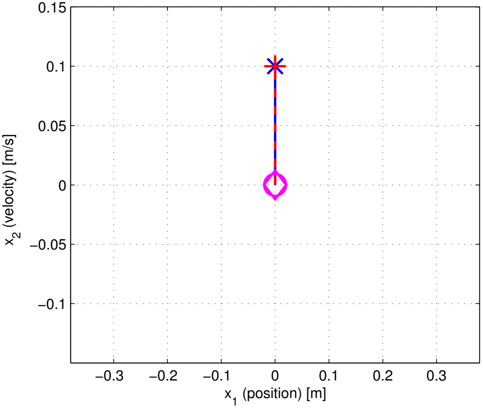
dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Naive estimate using *perfect* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \qquad \boxed{\hat{x}_0 = x_0}$$



x versus $\hat{x}$ at $t = 0^+$

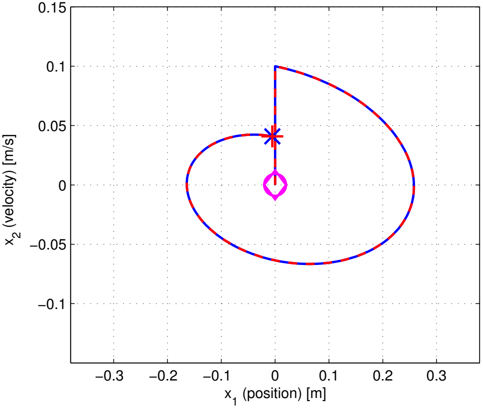
dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Naive estimate using *perfect* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \boxed{\hat{x}_0 = x_0}$$



x versus $\hat{x}$ at $t = 20$

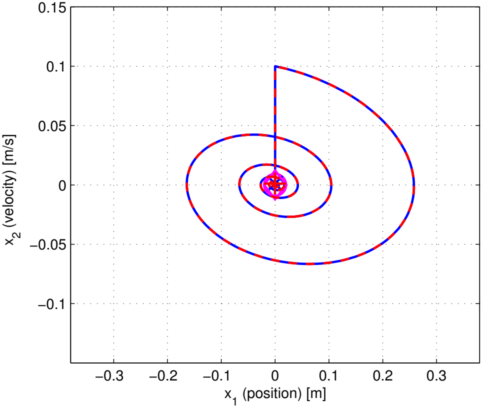
dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Naive estimate using *perfect* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \boxed{\hat{x}_0 = x_0}$$



x versus $\hat{x}$ at $t = 100$

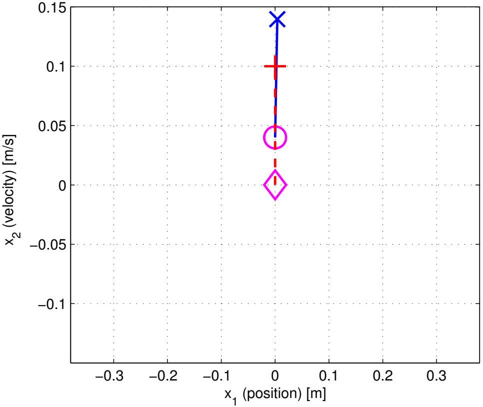
dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Naive estimate using *wrong* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \boxed{\hat{x}_0 \neq x_0}$$



x versus $\hat{x}$ at $t = 0^+$

dave.zachariah@it.uu.se

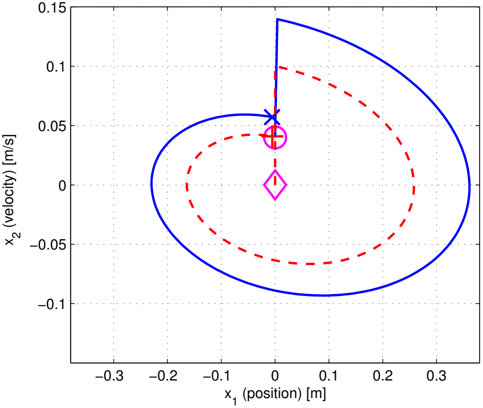
UPPSALA UNIVERSITET

Build intuition from simple systems

State estimation via simulation

Naive estimate using *wrong* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \qquad \boxed{\hat{x}_0 \neq x_0}$$



x versus $\hat{x}$ at $t = 20$

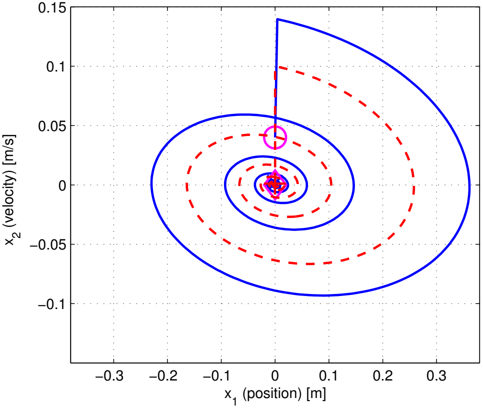
dave.zachariah@it.uu.se

Build intuition from simple systems

State estimation via simulation

Naive estimate using *wrong* initial guess:

$$\dot{\hat{x}} = A\hat{x} + Bu, \qquad \boxed{\hat{x}_0 \neq x_0}$$



x versus $\hat{x}$ at $t = 100$

dave.zachariah@it.uu.se

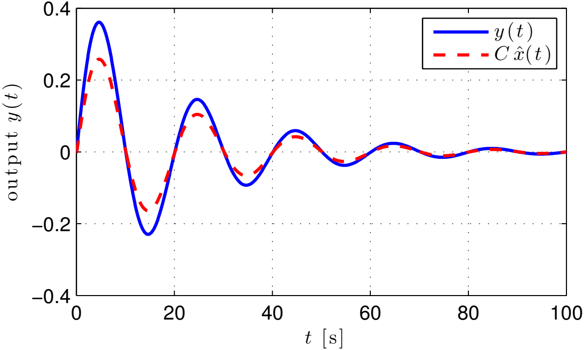
UPPSALA UNIVERSITET

Build intuition from simple systems

State estimation via simulation

x och $\hat{x}$ correspond to different outputs:

$$y = Cx \quad \text{versus} \quad \hat{y} = C\hat{x}$$



dave.zachariah@it.uu.se

Estimating the states via observer

dave.zachariah@it.uu.se

Estimating the states

Correcting the state estimates

- *Idea*: Feedback the prediction error $y - C\hat{x}$ to correct $\hat{x}$
- Observer: an estimator with a *correction term*

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K\big(y - C\hat{x}\big)}_{\text{correction}}, \quad \hat{x}(0) = \hat{x}_0$$

dave.zachariah@it.uu.se

UPPSALA UNIVERSITET

Estimating the states

Correcting the state estimates

- *Idea*: Feedback the prediction error $y - C\hat{x}$ to correct $\hat{x}$
- Observer: an estimator with a *correction term*

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K\big(y - C\hat{x}\big)}_{\text{correction}}, \quad \hat{x}(0) = \hat{x}_0$$

- Using matrix

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

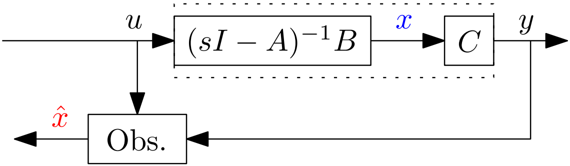
we can *design* the estimator.

dave.zachariah@it.uu.se

Estimating the states
Correcting the state estimates

- *Idea*: Feedback the prediction error $y - C\hat{x}$ to correct $\hat{x}$
- Observer: an estimator with a *correction term*

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K\left(y - C\hat{x}\right)}_{\text{correction}}, \quad \hat{x}(0) = \hat{x}_0$$



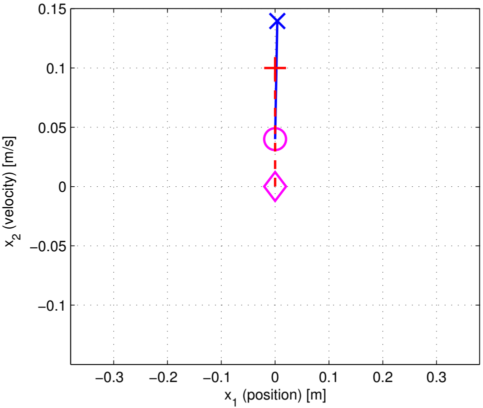
dave.zachariah@it.uu.se

Build intuition using simple systems

State estimation using observer

Estimation using observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + K\big(y - C\hat{x}\big), \quad \boxed{\hat{x}_0 \neq x_0}$$



x versus $\hat{x}$ at $t = 0^+$

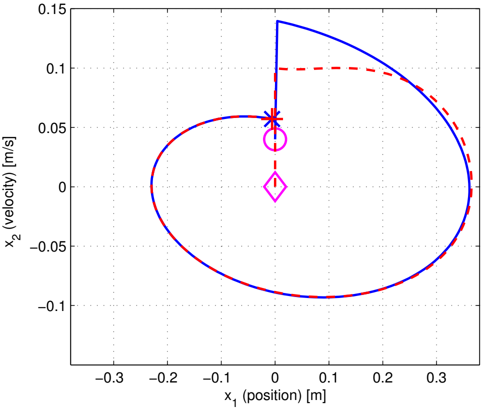
dave.zachariah@it.uu.se

Build intuition using simple systems

State estimation using observer

Estimation using observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + K\big(y - C\hat{x}\big), \qquad \boxed{\hat{x}_0 \neq x_0}$$



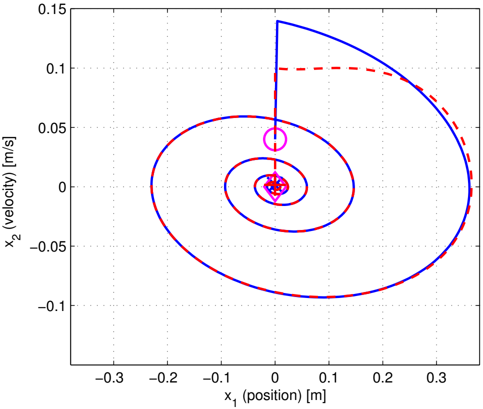
x versus $\hat{x}$ at $t = 20$

Build intuition using simple systems

State estimation using observer

Estimation using observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + K\big(y - C\hat{x}\big), \qquad \boxed{\hat{x}_0 \neq x_0}$$



x versus $\hat{x}$ at $t = 100$

dave.zachariah@it.uu.se

State estimation
Estimation error and observability

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

[Board: derive evolution of estimation errors]

dave.zachariah@it.uu.se

State estimation

Estimation error and observability

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

[Board: derive evolution of estimation errors]

Result

Errors of observer described as system

$$\tilde{x}(t) = e^{(A-KC)t}\tilde{x}(0)$$

and therefore $\|\tilde{x}(t)\|$ decays at a rate given by maximum

$$\text{Re}\{\tilde{s}_i\}$$

where $\tilde{s}_i$ are observer poles/eigenvalues of $(A - KC)$.

dave.zachariah@it.uu.se

State estimation

Estimation error and observability

Estimation error:

$$\boxed{\tilde{x} \triangleq x - \hat{x}}$$

Result 9.2

State-space form is observable (cf. $\det \mathcal{O} \neq 0$) $\Rightarrow$ matrix K can be chosen such that $\tilde{x}$ vanish arbitrarily quick

dave.zachariah@it.uu.se

State estimation
Estimation error and observability

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

Result 9.2

State-space form is observable (cf. $\det \mathcal{O} \neq 0$) $\Rightarrow$ matrix K can be chosen such that $\tilde{x}$ vanish arbitrarily quick

- K is solved by polynomial $\det(sI - A + KC) = 0$ with desired roots in left halfplane

dave.zachariah@it.uu.se

State estimation
Estimation error and observability

Estimation error:

$$\boxed{\tilde{x} \triangleq x - \hat{x}}$$

Result 9.2

State-space form is observable (cf. $\det \mathcal{O} \neq 0$) $\Rightarrow$ matrix K can be chosen such that $\tilde{x}$ vanish arbitrarily quick

- K is solved by polynomial $\det(sI - A + KC) = 0$ with desired roots in left halfplane
- Quick observer $\hat{x}$ is however sensitive to measurement noise!

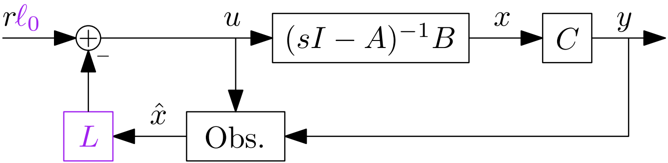
dave.zachariah@it.uu.se

Feedback with estimated states

dave.zachariah@it.uu.se

Feedback using estimated states

Controller the Laplace domain



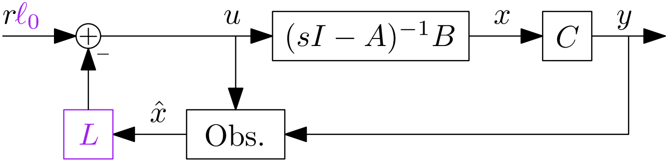
System and controller with observer:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \quad \text{and} \quad \begin{cases} u &= -L\hat{x} + \ell_0 r \\ \dot{\hat{x}} &= A\hat{x} + Bu + K(y - C\hat{x}) \end{cases}$$

dave.zachariah@it.uu.se

Feedback using estimated states

Controller the Laplace domain



Controller with observer:

$$\mathcal{L}: \begin{cases} U(s) &= -L\widehat{X}(s) + \ell_0 R(s) \\ s\widehat{X}(s) &= A\widehat{X}(s) + BU(s) + K\big(Y(s) - C\widehat{X}(s)\big) \end{cases}$$

[Board: solve for controller]

dave.zachariah@it.uu.se

Feedback using estimated states

General linear feedback control

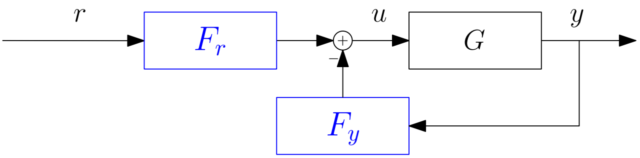
General linear feedback form, ch.9.5 G&L

Controller with observer can be written as

$$U(s) = F_r(s)R(s) - F_y(s)Y(s),$$

where

$$F_r(s) = (1 - L(sI - A + KC + BL)^{-1}B)\ell_0$$
$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$



dave.zachariah@it.uu.se

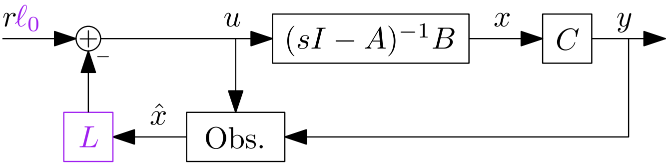
UPPSALA
UNIVERSITET

Resulting closed-loop system

dave.zachariah@it.uu.se

Feedback using estimated states

Effect of estimation error



Study system and controller with observer:

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \quad \text{and} \quad u = -L\hat{x} + \ell_0 r$$

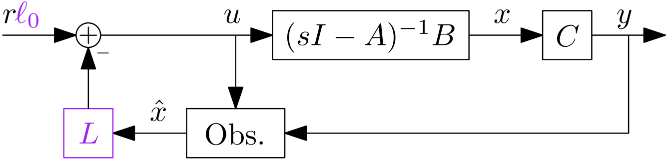
by substituting $\hat{x} = x - \tilde{x}$

[Board: derive the closed-loop system with estimation error $\tilde{x}$]

dave.zachariah@it.uu.se

Feedback using estimated states

Effect of estimation error



Yields closed-loop system:

$$\dot{x} = (A-BL)x + \overbrace{BL\tilde{x}}^{\text{effect of estimation error}} + B\ell_0 r$$
$$y = Cx$$

with additional error states

$$\dot{\tilde{x}} = (A-KC)\tilde{x} \quad \to \; 0$$

[Board: write the closed-loop system in state-space form]

dave.zachariah@it.uu.se

Feedback using estimated states

The closed-loop system with observer

The closed-loop system with estimation error can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}}_{\widetilde{A}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\widetilde{B}} \ell_0 \, r$$

$$y = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{\widetilde{C}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

with extended state vector.

This yields transfer function from r to y:

$$\Rightarrow G_c(s) = \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$$

dave.zachariah@it.uu.se

Feedback using estimated states

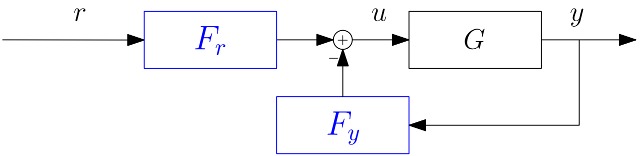
The closed-loop system with observer

Closed-loop system transfer function, ch.9.5 G&L

Insert matrices $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ yields

$$G_c(s) = \widetilde{C}(sI - \widetilde{A})^{-1}\widetilde{B}$$
$$= C(sI - A + BL)^{-1}B\ell_0$$

with same poles as if states were known and K is gone!



dave.zachariah@it.uu.se

Summary and recap

- Rules of thumb for pole placement
- Estimation using observer
- Feedback using estimated states
- Closed-loop system with observer

dave.zachariah@it.uu.se