Introduction to Computer Control Systems, 5 credits, 1RT485

Date and Time: 2016-03-14

Place: Polacksbacken, skrivsalen.

Teacher on duty: Dave Zachariah.

Allowed aid:

- A basic calculator
- BETA mathematical handbook

NB: Only one problem per sheet. Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Solutions have to be explained in detail.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Table 2: Properties of Laplace Transforms

$\mathcal{L}\left[af(t)\right] = aF(s)$	$\mathcal{L}\left[tf(t)\right] = -\frac{dF(s)}{ds}$
$\mathcal{L}\left[f_1(t) \pm f_2(t)\right] = F_1(s) \pm F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$	$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n}F(s); n = 1,2,3,...$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - f'(0)$	$\mathcal{L}\left[f(\frac{t}{a})\right] = aF(as)$
$\mathcal{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)dt\right]\Big\|_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a)$	

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1 - S(s)$$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- $$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

We can write the following control structures in Laplace form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K \left(\frac{\tau_D s + 1}{\beta \tau_D s + 1} \right) \left(\frac{\tau_I s + 1}{\tau_I s + \gamma} \right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \big(1 - L(sI - A + KC + BL)^{-1}B\big)\, \ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = [\text{if } A^{-1} \text{ exists}] = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 0: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) The following system is controllable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ii) The system

$$G(s) = \frac{s+2}{(s+1)(s^2 - s + 20)}$$

is input-output stable.

iii) Suppose the frequency response of a closed-loop system gives $|G_c(i2\pi 10)| = 2$. Using a reference signal $r(t) = 2\sin(2\pi 10t)$, yields an output $y(t)$ with an amplitude 4 (after transients have vanished).

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) Given the sensitivity functions $S(s)$ and $T(s)$ of a system, the Nyquist contour is given by $G_o(i\omega) = T(i\omega)/S(i\omega)$

ii) Given an observable system, the poles of its closed-loop system using a state-feedback controller can always be placed arbitrarily.

iii) If the eigenvalues of a system matrix A are $-3 \pm i3$ and 1, we can guarantee that the system from u to y is unstable.

(3 p)

Proposed solution to problem 0

Part A

i) False.

The controllability matrix equals

$$\mathcal{S} = [B\ AB] = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$$

for which $\det(\mathcal{S}) = 0$.

ii) False.

The poles are $s = -1, 4$ and 5.

iii) True.

From sine-in/sine-out, the output is $y(t) = |G_c(i2\pi\omega)| \cdot A \sin(\omega t + \arg\{G_c(i2\pi\omega)\})$ when $r(t) = A\sin(\omega t)$

Part B

i) True.

Using the fact that $T = G_0/(1 + G_0)$

ii) False.

The system must be controllable.

iii) False.

In general the poles of the system is a subset of the eigenvalues of A.

Problem 1 (6/30)

a) Consider a system $G_q(s)$ that converts a voltage $u(t)$ to a force $w(t)$. The force acts on a spring damper modeled as $G_d(s)$ with position $y(t)$. The complete system model is illustrated as

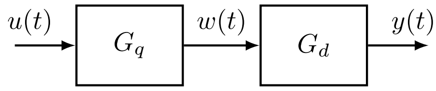


Figure 1: System from voltage $u(t)$ to position of spring damper $y(t)$.

and can be written as

$$Y(s) = G_d(s)G_q(s)U(s) = \frac{s+4}{s^2 - 2s + 2}U(s).$$

Using P-controller $F(s) = K$ we construct an active damper. Is the closed-loop system $G_c(s)$ stable when $K = 3$?

(4 p)

b) One model for $G_q(s)$ is

$$G_q(s) = \frac{s^2 - 1}{(s+1)(s^2 + 5s + 6)}$$

Is system is minimum phase or not?

(2 p)

Proposed solution to problem 1

a) Closed-loop system:

$$G_c = \frac{GF}{1+GF} = \frac{\frac{s+4}{s^2-2s+2}K}{1+\frac{s+4}{s^2-2s+2}K} = \frac{(s+4)3}{s^2-2s+2+(s+4)3} = \frac{3s+12}{s^2+s+14}$$

Stability can be checked using Routh's algorithm or computing poles:

$$s = -\frac{1}{2} \pm \frac{\sqrt{55}}{2} i$$

which are strictly in the left half-plane.
Therefore the system is input-output stable.

b) Minimum phase systems have neither poles nor zeros in the right half-plane (or time-delays).
The system can first be written on factored form

$$G_q(s) = \frac{(s+1)(s-1)}{(s+1)(s+2)(s+3)} = \frac{s-1}{(s+2)(s+3)}.$$

The resulting zero is located at $+1$ and the system is non-minimum phase

Problem 2 (6/30)

a) Consider a controlling a chemical process with an output modeled as

$$Y(s) = G(s)U(s),$$

where the process model can be written as

$$G(s) = \frac{s+3}{s^2+6s+11}.$$

Using a state-feedback controller, assuming that the states can be obtained, design a stable closed-loop system from r to y with non-oscillating poles located at -1 and -2.

(3 p)

b) Now suppose the states of the chemical process cannot be obtained but are estimated using an observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}).$$

where

$$K = \begin{bmatrix} 28 \\ 58 \end{bmatrix}$$

Combining the observer with the state-feedback controller results in a new closed-loop system from r to y. Derive its transfer function $G_c(s)$.

(2 p)

c) Compute the poles of $G_c(s)$

(1 p)

Proposed solution to problem 2

a) The system can be written on controllable canonical form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-6 & -11\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} u$$
$$y = \begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Using a state-feedback controller

$$u = -Lx + \ell_0 r$$

yields a closed-loop system with poles:

$$\det(sI - A + BL) = 0.$$

and the desired polynomial is $(s+1)(s+2) = s^2 + 3s + 2$ which we can now match.
Then

$$sI - A + BL = ... = \begin{bmatrix}s + 6 + \ell_1 & 11 + \ell_2\\ -1 & s\end{bmatrix}$$

where ℓ_i are on the top row and the determinant is easy to compute in as result of the controllable canonical form:

$$\det(sI - A + BL) = s^2 + (6 + \ell_1)s + (11 + \ell_2).$$

Identification yields

$$\begin{cases} 6 + \ell_1 = 3 \\ 11 + \ell_2 = 2 \end{cases} \quad \Rightarrow \begin{cases} \ell_1 = 3 - 6) \\ \ell_2 = 2 - 11 \end{cases} \quad \Rightarrow L = \begin{bmatrix}-3\\ -9\end{bmatrix}$$

b) Note that the closed-loop system is the same as without an observer, that is,

$$G_c(s) = C(sI - A + BL)^{-1}B\ell_0.$$

From above we have that

$$(sI - A + BL)^{-1} = \begin{bmatrix}s+3 & 2\\ -1 & s\end{bmatrix}^{-1} = \frac{1}{(s+3)s+2}\begin{bmatrix}s & -2\\ 1 & s+3\end{bmatrix}.$$

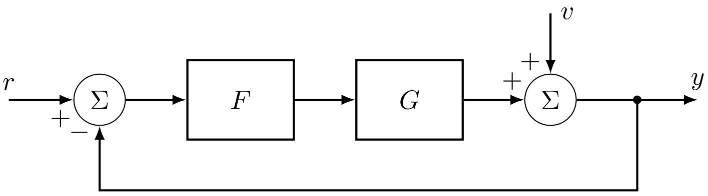
Then

$$\begin{aligned} G_c(s) &= C(sI - A + BL)^{-1}B\ell_0 \\ &= \begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}s & -2\\ 1 & s+3\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}\ell_0\frac{1}{(s+3)s+2} \\ &= \begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix}s\\ 1\end{bmatrix}\ell_0\frac{1}{(s+3)s+2} \\ &= \frac{(s+3)\ell_0}{s^2+3s+2} \end{aligned} \tag{1}$$

c) Poles are given by pole placement: -1 and -2!
Compare $s^2 + 3s + 2 = 0$

Problem 3 (6/30)

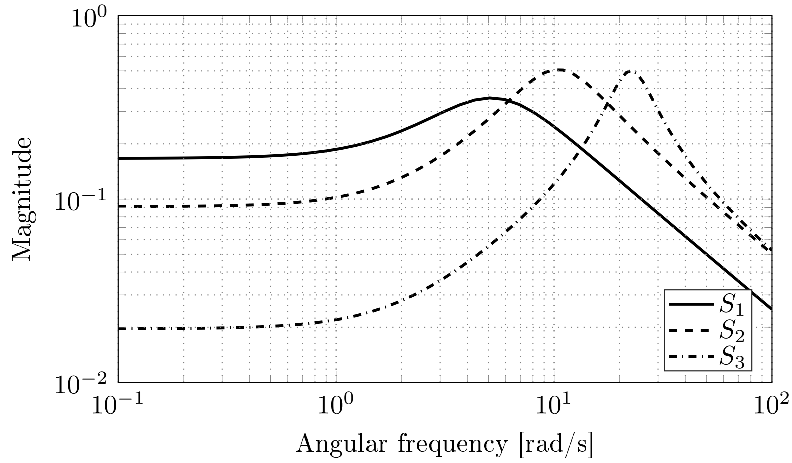
a) Study the feedback system below



where

$$G(s) = \frac{s}{s+2}.$$

Assume that $v(t)$ is not measurable but has most of its energy concentrated around 10 rad/s. The Bode plot below shows the sensitivity function $S = \frac{1}{1+GF}$ for the system with three different choices of the controller F. Which controller is best at suppressing the disturbance?

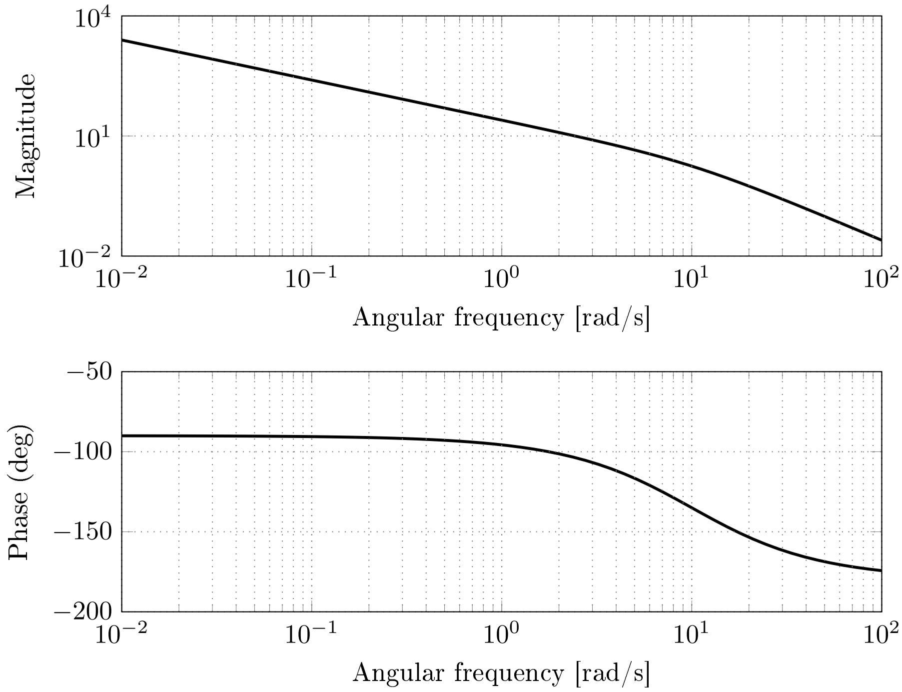


(3 p)

b) When the disturbance $v(t)$ can be measured, we want to use this for control. Draw a block diagram of such a controller and design it such that $v(t)$ does not affect the output signal $y(t)$.

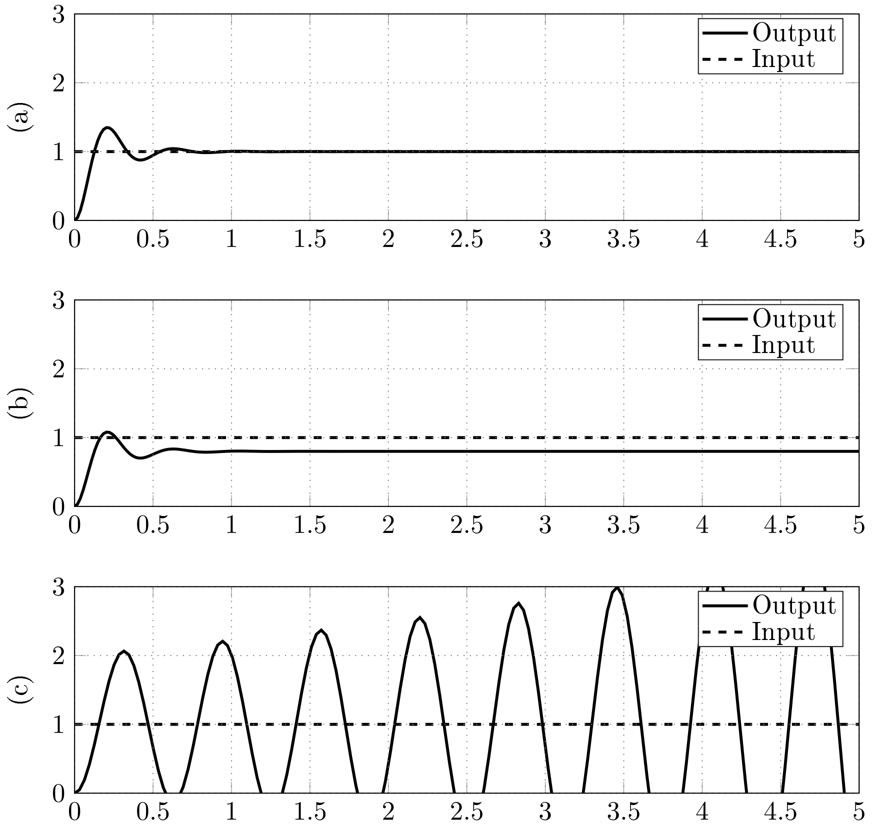
(2 p)

c) The Bode diagram for an open system G is given below.



The system is put in a feedback loop with a proportional controller with unit gain, i.e. $K = 1$. The figure below shows three step responses (a)-(c). Which step response could come from the closed loop system?

Hint: Use the properties of the Nyquist contour $G_o(i\omega)$.


(a)
Output
Input
(b)
Output
Input
(c)
Output
Input
0
1
2
3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

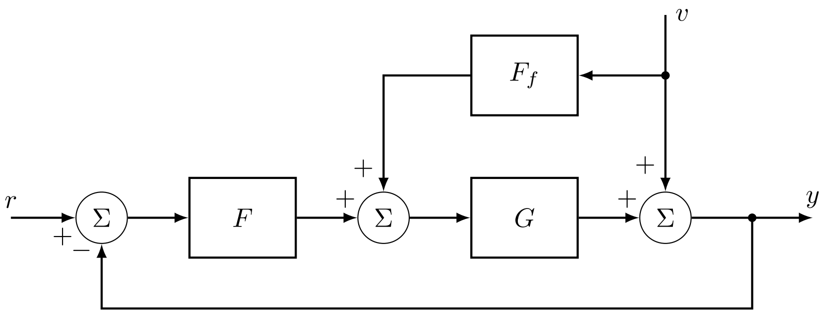
(1 p)

Proposed solution to problem 3

a) The transfer function from the disturbance to the output is given by the sensitivity function $S(s)$. Controller 3 is to prefer since it gives the lowest magnitude for the sensitivity function at 10 rad/s.

Answer: Controller 3.

b) Use feed forward control from the disturbance as in the block diagram below.



To eliminate the effects of the disturbance on the output we have to design F_f. From the block diagram we get the transfer function from v to y as

$$Y(s) = V(s) + G(s)\left(F_f(s)V(s) - F(s)Y(s)\right) \implies Y(s) = \frac{1 + G(s)F_f(s)}{1 + G(s)F(s)}V(s).$$

Having $1 + G(s)F_f(s) = 0$ would eliminate the disturbance. Hence, we get

$$F_f(s) = -\frac{1}{G(s)} = -\frac{s+2}{s}.$$

Answer: $F_f(s) = -\frac{s+2}{s}$.

c) The closed loop system will be stable since Nyquist contour $G_o(i\omega) = G(i\omega)F(i\omega) = G(i\omega) \cdot 1$ has positive phase margin $\varphi_m = \arg\{G_o(i\omega_c)\} + 180 > 0$, where ω_c is the crossover frequency at which $|G_o(i\omega_c)| = 1$. Hence step response (c) is eliminated.

It is also reasonable to believe that the system has infinite static gain $|G_o(0)| = \infty$ (the slope in the Bode diagram for low frequencies is -1), which will give the closed loop system $|G_c(0)| = |G_o(0)|/|1 + G_o(0)| = 1$. By this we can eliminate (b).

Answer: (a)

Problem 4 (6/30)

A minesweeper vessel uses hydrophones in order to detect underwater mines. In particular, the hydrophones are arranged in an array in order to cover a larger terrain. The transfer function from the drive shaft which is connected to the vessel to the hydrophone array can be given as a second order system:

$$G(s) = \frac{1}{Js^2 + k_d s}$$

where J is the moment of inertia of the array and k_d represents the viscous force of the water. The transfer function above can be rewritten as a state-space representation by using e.g. the controller canonical form:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} -\frac{k_d}{J} & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & \frac{1}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

where u is the input from the drive shaft and y is the output to the array.

We want to control the array with a computer.

a) Discretize the state-space system above with a sampling time T

(3 p)

b) Assume that $J = 2\ \mathrm{Nm\,s^2\,rad^{-1}}$, $k_d = 8\ \mathrm{Nm\,s\,rad^{-1}}$ and $T = \frac{1}{4}$ s. By using state feedback on the discrete-time system, compute a suitable matrix $\mathbf{L}$ so that the system error will decay as e^{-t}, i.e. the poles in the continuous-time system are at -1.

Note: Answers in rounded decimals are fine.

(3 p)

Proposed solution to problem 4

a) To do so, we use that for a ZOH the discretized system is given by:

$$F = e^{AT} \quad G = \int_0^T e^{A\tau} B \, \mathrm{d}\tau$$

and the matrix exponential can be computed as:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

The inverse of $(sI - A)$ is given by:

$$(sI - A)^{-1} = \begin{bmatrix} s + \frac{k_d}{J} & 0 \\ -1 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+\frac{k_d}{J}} & 0 \\ \frac{1}{s\left(s+\frac{k_d}{J}\right)} & \frac{1}{s} \end{bmatrix}$$

Taking the inverse Laplace transform:

$$F = e^{AT} = \begin{bmatrix} e^{-\frac{k_d}{J}T} & 0 \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}T}\right) & 1 \end{bmatrix}$$

Once this is known, computing G is straightforward taking into account that the integral operator can go inside each component of the resulting matrix:

$$\begin{aligned} G &= \int_0^T \begin{bmatrix} e^{-\frac{k_d}{J}\tau} & 0 \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}\tau}\right) & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{0} \end{bmatrix} \mathrm{d}\tau \\ &= \int_0^T \begin{bmatrix} e^{-\frac{k_d}{J}\tau} \\ \frac{J}{k_d}\left(1 - e^{-\frac{k_d}{J}\tau}\right) \end{bmatrix} \mathrm{d}\tau \\ &= \begin{bmatrix} -\frac{J}{k_d}\left(e^{-\frac{k_d}{J}T} - 1\right) \\ \frac{J}{k_d}\left(T + \frac{J}{k_d}e^{-\frac{k_d}{J}T} - \frac{J}{k_d}\right) \end{bmatrix} \end{aligned}$$

and $H = C$.

b) Substituting:

$$F = \begin{bmatrix} 0.37 & 0 \\ 0.16 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.16 \\ 0.023 \end{bmatrix}$$

Using state feedback control ($u = -Lx$), the matrix which gives the system dynamics in closed loop is given by $F - GL$:

$$F - GL = \begin{bmatrix} 0.37 - 0.16\ell_1 & -0.16\ell_2 \\ 0.16 - 0.023\ell_1 & 1 - 0.023\ell_2 \end{bmatrix}$$

In order to perform pole placement, the characteristic polynomial of $F - GL$ (which will give the eigenvalues λ) must be computed and then made equal to the desired characteristic polynomial:

$$\det(\lambda I - (F - GL)) = \left\| \begin{array}{cc} \lambda - 0.37 + 0.16\ell_1 & 0.16\ell_2 \\ -0.16 + 0.023\ell_1 & \lambda - 1 + 0.023\ell_2 \end{array} \right\| =$$

$$= (\lambda - 0.37 + 0.16\ell_1)(\lambda - 1 + 0.023\ell_2) - 0.16\ell_2(-0.16 + 0.023\ell_1)$$

$$= \lambda^2 + (-1.37 + 0.023\ell_2 + 0.16\ell_1)\lambda + 0.37 + 0.017\ell_2 - 0.16\ell_1 = 0$$

The desired poles are at -1 in continuous time. They must be translated into discrete time with the expression $p_d = e^{p_c T}$, where p_d is the pole in discrete time and p_c is the pole in continuous time. Doing so, the desired characteristic polynomial is $(\lambda - e^{-1/4})^2 = \lambda^2 - 2e^{-1/4}\lambda + e^{-1/2} = 0$. Since we want this polynomial to be the same as the one obtained above for the system, the coefficients of both polynomials must be equal, obtaining the following system of equations:

$$\begin{array}{rrrl} \lambda: & 0.16\ell_1 & +0.023\ell_2 & = -0.187 \\ 1: & -0.16\ell_1 & +0.017\ell_2 & = 0.236 \end{array}$$

which once solved yield the values $\ell_1 = -1.345$ and $\ell_2 = 1.225$.