Introduction to Computer Control Systems, 5 credits, 1RT485

Date and Time: 2016-06-10

Place: Polacksbacken, skrivsalen.

Teacher on duty: Dave Zachariah.

Allowed aid:

- A basic calculator
- BETA mathematical handbook

NB: Only one problem per sheet. Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Solutions have to be explained in detail.

Best of luck!

Useful results

Laplace transform table

Table 1: Basic Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit impulse $\delta(t)$	1	$\sinh(bt)$	$\frac{b}{s^2-b^2}$
unit step $1(t)$	$\frac{1}{s}$	$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t	$\frac{1}{s^2}$	$\frac{1}{2b}t\sin(bt)$	$\frac{s}{(s^2+b^2)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$	$\frac{s}{(s^2+a^2)(s^2+b^2)}$
$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{s^2}{(s^2+a^2)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n=1,2,3...)$	$\frac{1}{(s+a)^n}$		
$\sin(bt)$	$\frac{b}{s^2+b^2}$		
$\cos(bt)$	$\frac{s}{s^2+b^2}$		
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$		
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$		

Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1 - S(s)$$

Table 2: Properties of Laplace Transforms

$\mathcal{L}[af(t)] = aF(s)$	$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$
$\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$	$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$
$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$	$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n=1,2,3,\ldots$
$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$	$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$
$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}+\frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$	$\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$
$\mathcal{L}[f(t-a)] = e^{-as}F(s)$	$\mathcal{L}\left[e^{-at}f(t)\right] = F(s+a)$

State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau) d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K \left(\frac{\tau_D s + 1}{\beta \tau_D s + 1} \right) \left(\frac{\tau_I s + 1}{\tau_I s + \gamma} \right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$F_r(s) = \big(1 - L(sI - A + KC + BL)^{-1}B\big)\,\ell_0$$
$$F_y(s) = L(sI - A + KC + BL)^{-1}K$$

Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period T can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = \lceil \text{if } A^{-1} \text{ exists} \rceil = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives -1 point. Minimum total points for Part A and B is 0 , respectively.

Part A

Note: Write 'skip' if your total home assignment score ≥ 8

i) The following system is observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

ii) The system

$$G(s) = \frac{s^2 - 9}{(s-3)(s^2 + 8s + 12)}$$

is input-output stable.

iii) The output of a closed-loop system is $y(t) = 4\sin(2\pi 50t + \pi/16)$ when the reference signal $r(t)$ is a sinusoid with frequency $\omega = 2\pi 50$ and amplitude 2. Therefore magnitude $|G_c(i\omega)| = 2$ at frequency $\omega = 2\pi 50$.

(3 p)

Part B

Note: Write 'skip' if your total home assignment score ≥ 12

i) The complementary sensitivity function of a system can be alternatively be computed $T(s) = S(s)G_o(s)$ of a system.

ii) Given an observable system, the errors of an observer which tracks the states can be made to vanish arbitrarily fast.

iii) Suppose a system has 3×3 system matrix A and a transfer function

$$G(s) = \frac{s+1}{(s+2)(s+5)}.$$

Then we know the eigenvalues of A equal -2 and -5.

(3 p)

Proposed solution to problem 1

Part A

i) False.

Compute determinant of observability matrix.

ii) True.

All poles are in the left-halfplane

iii) True.

Use sine-in/sine-out relatin.

Part B

i) True.

Apply definitions of the functions.

ii) True.

Observer poles can then be placed arbitrarily.

iii) False.

In general the poles of the system is a subset of the eigenvalues of A.

Problem 2 (6/30)

a) Consider a mechanical system $G_q(s)$ that converts a force $u(t)$ into a torque $w(t)$. Another mechanical system $G_d(s)$ converts the torque on the position of an arm $y(t)$ that we wish to control. The complete system model is illustrated as

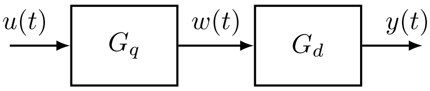


Figure 1: System from voltage $u(t)$ to position of spring damper $y(t)$.

and can be written as

$$Y(s) = G_d(s)G_q(s)U(s) = \frac{2}{s^2 + 5s - 10}U(s).$$

Design a stable closed-loop system $G_c(s)$ using a P-controller $F(s) = K$.

(3 p)

b) Let the reference signal $r(t)$ be a step. That is,

$$r(t) = \begin{cases} r_0, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Suppose K_0 is a setting of the P-controller that yields a stable closed-loop system. Calculate the stationary error of the controller: $e(t) = r(t) - y(t)$ as $t \to \infty$.

(3 p)

Proposed solution to problem 2

a) Compute closed-loop system:

$$G_c = \frac{GK}{1+GK}$$

Stability conditions for K be determined using Routh's algorithm or computing poles. Then K can be chosen accordingly.

b) Note that the error can be written in the Laplace domain as

$$E = R - Y = \left(1 - \frac{GK}{1+GK}\right) R.$$

Use this along with the final value theorem to obtain

$$\lim_{t \to \infty} e(t)$$

when the closed-loop system is stable.

Problem 3 (6/30)

a) Consider a controlling a chemical process with an output modeled as

$$Y(s) = G(s)U(s),$$

where the process model can be written as

$$G(s) = \frac{s+3}{s^2+6s+11}.$$

We wish to estimate the states x of the system. Design an observer $\hat{x}$ that achieves this with observer poles located at -1 and -2.

(4 p)

c) At what rate does the magnitude of estimation errors $\tilde{x}(t) = x(t) - \hat{x}(t)$ decay if the observer poles are as above?

(2 p)

Proposed solution to problem 3

a) An observer can be written mathematically as

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}),$$

where the matrix K has to be designed.
The state errors then evolve as

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

and the eigenvalues/poles of this system is therefore given by the roots of $\det(sI - A - KC) = 0$, which we can design by choosing K. By setting

b) Since the errors evolve according to

$$\dot{\tilde{x}}(t) = (A - KC)\tilde{x}(t)$$

we know the rate of decay is given by the eigenvalues/poles. The maximum eigenvalue λ give rise to the slowest decay $e^{\lambda t}$. Since we have designed the observers with eigenvalues -1 and -2 the decay is e^{-1t}

Problem 4 (6/30)

The Bode diagram for an industrial process is shown in Figure 2.

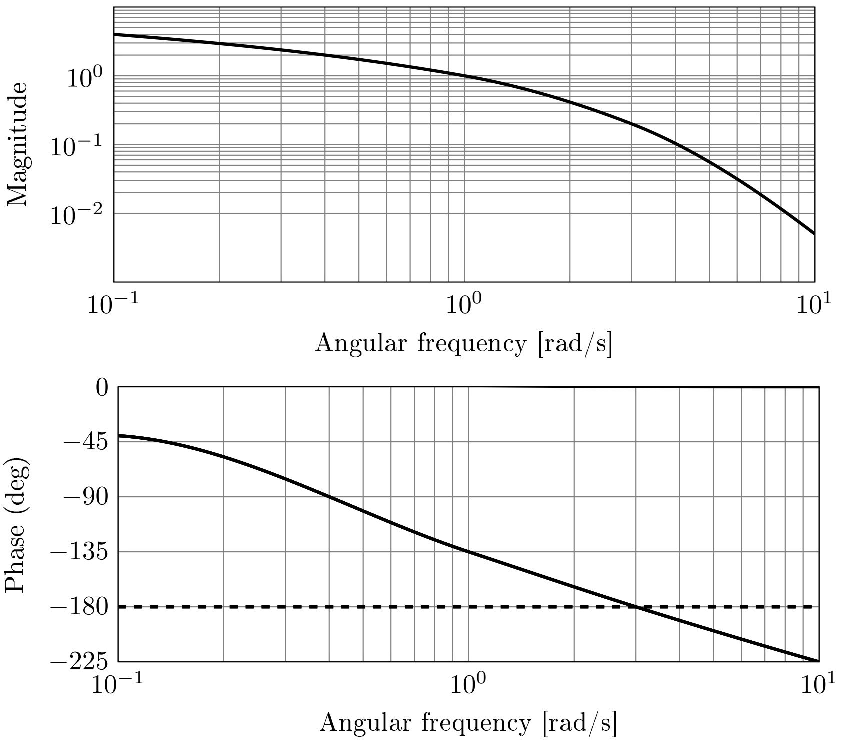


Figure 2: Bode diagram for problem 4

(a) What is the crossover frequency ω_c and the phase margin φ_m for the process?

(1 p)

(b) Suppose a pure time delay is introduced in the system, that is we measure $\tilde{y}(t) = y(t - \tau)$ instead of $y(t)$. What is the greatest value of the time delay τ before the process becomes unstable if we use a simple feedback controller with $F(s) = 1$?

Hint: See Table 2 for properties of the Laplace transform.

(2 p)

(c) Design a PD-controller $F(s) = K(1 + T_d s)$ such that the controlled system gets a crossover frequency ω_c three times higher than the uncontrolled system without time delay but having the same phase margin φ_m.

(3 p)

Proposed solution to problem 4

a) The crossover frequency ω_c is given by the frequency where the amplitude curve goes below 1 and the phase margin ϕ_m is given be the distance between -180 and the phase curve at the crossover frequency.

Answer: $\omega_c = 1$ rad/s, $\phi_m = 45°$.

b) The system with the time delay is given by $\tilde{G}(s) = e^{-\tau s}G(s)$. The closed loop system is stable if the loop gain has positive phase margin. Since the time delay does not affect the gain,

$$|\tilde{G}(i\omega)| = |e^{-i\tau\omega}G(i\omega)| = \underbrace{|e^{-i\tau\omega}|}_{=1}|G(i\omega)| = |G(i\omega)|,$$

the requirement is that

$$\arg\tilde{G}(i\omega_c) = \arg e^{-i\tau\omega_c}G(i\omega_c) = \arg e^{-i\tau\omega_c} + \arg G(i\omega_c) = -\tau\omega_c - 135° > -180°$$

$$\Leftrightarrow$$

$$\tau < 45\frac{\pi}{180}\frac{1}{\omega_c} = 0.79\,\text{s}$$

Answer: $\tau < 0.79$ s.

c) The desired crossover frequency $\omega_{c,d} = 3$ rad/s and the desired phase margin $\phi_{m,d} = 45°$. This gives us two equations. For the gain we get

$$1 = |G(i\omega_{c,d})F(i\omega_{c,d})| = \underbrace{|G(i\omega_{c,d})|}_{\text{from Figure 2}}|K(1 + T_d i\omega_{c,d)}| = 0.2K\sqrt{1 + T_d^2\omega_{c,d}^2}$$

and for the phase we get

$$-135° = \arg G(i\omega_{c,d})F(i\omega_{c,d}) = \underbrace{\arg G(i\omega_{c,d})}_{\text{from Figure 2}} + \arg F(i\omega_{c,d}) = -180° + \arctan T_d\,\omega_{c,d}$$

The second equation directly gives

$$T_d = \frac{1}{\omega_{c,d}}\tan(45°) = \frac{1}{3}\cdot 1 \approx 0.33,$$

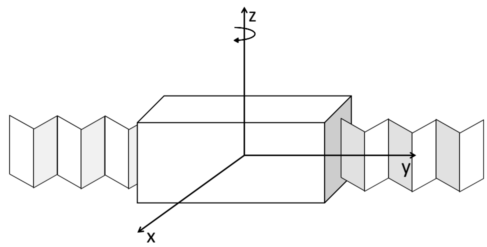
which inserted into the first equation gives

$$K = \frac{1}{0.2\sqrt{1 + T_d^2 w_{c,d}^2}} = \frac{5}{\sqrt{1 + \frac{1}{3^2}3^2}} = \frac{5}{\sqrt{2}} \approx 3.54.$$

Answer: $K = 3.54$ and $T_d = 0.33$.

Problem 5 (6/30)

A satellite is orbiting the Earth. A simple way of modeling a rotation with respect to one of its axis is to use rigid-body mechanics.



In absence of other forces, a rotation given by an external force with torque τ can be expressed as:

$$\tau = I\alpha = I\ddot{\theta}$$

where I is the moment of inertia along the rotation axis and $\alpha = \ddot{\theta}$ is the angular acceleration. Assuming the torque as the input of the system, one can write the differential equation above as the following state-space model:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x_1 = \theta$ and $x_2 = \dot{\theta}$.

a) Discretize the state space model given above for a given sampling time T.

(3 p)

b) Suppose that we want to control the orientation of the satellite. We want the *discrete-time* system to have the poles at -0.5. Compute the state-feedback matrix. Use the values $T = 1$ s and $I = 0.1\ kg \cdot m^2$.

(3 p)

Proposed solution to problem 5

a) To do so, we use that for a ZOH the discretized system is given by:

$$F = e^{AT} \quad G = \int_0^T e^{A\tau} B d\tau$$

and the matrix exponential can be computed as:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

The inverse of $(sI - A)$ is given by:

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

Taking the inverse Laplace transform is straightforward, obtaining that:

$$F = e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Once this is known, computing G is straightforward taking into account that the integral operator can go inside each component of the resulting matrix:

$$G = \int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} = \int_0^T \begin{bmatrix} \frac{\tau}{I} \\ \frac{1}{I} \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2I} \\ \frac{T}{I} \end{bmatrix}$$

and $H = C$.

b) Substituting in F and G for the given values:

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

In order to use state feedback we build the closed-loop matrix (F-GL):

$$F - GL = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 10 \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 \end{bmatrix} = \begin{bmatrix} 1 - 5\ell_1 & 1 - 5\ell_2 \\ -10\ell_1 & 1 - 10\ell_2 \end{bmatrix}$$

Since the desired poles are specified (in this case, a double pole at -0.5), we'll use pole placement to compute ℓ_1 and ℓ_2. First, we compute the characteristic polynomial of $(F - GL)$:

$$\begin{aligned} \det(\lambda I - (F - GL)) &= \left\| \begin{matrix} 5\ell_1 - 1 + \lambda & 5\ell_2 - 1 \\ 10\ell_1 & 10\ell_2 - 1 + \lambda \end{matrix} \right\| \\ &= (5\ell_1 - 1 + \lambda)(10\ell_2 - 1 + \lambda) - (5\ell_2 - 1)10\ell_1 \\ &= \lambda^2 + (5\ell_1 + 10\ell_2 - 2)\lambda + 5\ell_1 - 10\ell_2 + 1 \end{aligned}$$

The desired characteristic polynomial is $(\lambda + 0.5)^2 = \lambda^2 + \lambda + 0.25$. Since both polynomials must be equal, the coefficients associated with them must be the same. Thus:

$$\begin{aligned} \lambda^2: &\quad 1 = 1 \\ \lambda: &\quad 5\ell_1 + 10\ell_2 = 3 \\ 1: &\quad 5\ell_1 - 10\ell_2 = -0.75 \end{aligned}$$

which yields the values of $\ell_1 = 0.225$ and $\ell_2 = 0.1875$.