# Introduction to Computer Control Systems, 5 credits, 1RT485

**Date and Time:** 2017-03-13

**Place:** Polacksbacken, skrivsalen.

**Teacher on duty:** Dave Zachariah.

**Allowed aid**:

- A basic calculator
- Beta mathematical handbook

**NB: Only one problem per sheet.** Write your anonymous exam code on each sheet. Write your name if you do not have an anonymous code.

Solutions have to be reproducible and explained in detail.

Best of luck!

# Useful results

## Laplace transform table

Table 1: Basic Laplace transforms

| $f(t)$ | $F(s)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| unit impulse $\delta(t)$ | $1$ | $\sinh(bt)$ | $\frac{b}{s^2-b^2}$ |
| unit step $1(t)$ | $\frac{1}{s}$ | $\cosh(bt)$ | $\frac{s}{s^2-b^2}$ |
| $t$ | $\frac{1}{s^2}$ | $\frac{1}{2b}t\sin(bt)$ | $\frac{s}{(s^2+b^2)^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $t\cos(bt)$ | $\frac{s^2-b^2}{(s^2+b^2)^2}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ | $\frac{\cos(bt)-\cos(at)}{a^2-b^2}; (a^2 \neq b^2)$ | $\frac{s}{(s^2+a^2)(s^2+b^2)}$ |
| $\frac{1}{a}(1-e^{-at})$ | $\frac{1}{s(s+a)}$ | $\frac{\sin(at)+at\cos(at)}{2a}$ | $\frac{s^2}{(s^2+a^2)^2}$ |
| $\frac{1}{(n-1)!}t^{n-1}e^{-at}; (n = 1, 2, 3...)$ | $\frac{1}{(s+a)^n}$ | | |
| $\sin(bt)$ | $\frac{b}{s^2+b^2}$ | | |
| $\cos(bt)$ | $\frac{s}{s^2+b^2}$ | | |
| $e^{-at}\sin(bt)$ | $\frac{b}{(s+a)^2+b^2}$ | | |
| $e^{-at}\cos(bt)$ | $\frac{s+a}{(s+a)^2+b^2}$ | | |

Table 2: Properties of Laplace Transforms

| | |
|---|---|
| $\mathcal{L}[af(t)] = aF(s)$ | $\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$ |
| $\mathcal{L}[f_1(t)+f_2(t)] = F_1(s)+F_2(s)$ | $\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2}{ds^2}F(s)$ |
| $\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s)-f(0)$ | $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s), \quad n = 1, 2, 3, \ldots$ |
| $\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s)-sf(0)-f'(0)$ | $\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$ |
| $\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\,dt\right]_{t=0}$ | $\mathcal{L}\left[\int_0^t f_1(t-\tau)f(t)d\tau\right] = F_1(s)F_2(s)$ |
| $\mathcal{L}[f(t-a)] = e^{-as}F(s)$ | $\mathcal{L}[e^{-at}f(t)] = F(s+a)$ |

## Matrix exponential

$$\boxed{e^{At} \triangleq \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}}$$

## Open-loop and sensitivity functions

$$G_o(s) = G(s)F_y(s), \qquad S(s) = \frac{1}{1+G_o(s)}, \qquad T(s) = 1-S(s)$$

## State-space forms and transfer function relations

- State-space form and transfer function

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \Rightarrow \quad \boxed{G(s) = C(sI - A)^{-1}B + D}$$

- Associated matrices

$$\mathcal{S} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \qquad \mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- LTI system with transfer function

$$\boxed{G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}}$$

i) Observable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & \cdots & 0 \\ -a_3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \\ \vdots \\ b_n - a_n b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

ii) Controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Solution to state-space equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

can be written as

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A\tau} Bu(t - \tau)d\tau}$$

- Observer system

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})}$$

## Feedback control structures

General linear feedback in Laplace form:

$$\boxed{U(s) = F_r(s)R(s) - F_y(s)Y(s)}$$

Common control structures in this form.

- PID controller:

$$F_y(s) = F_r(s) = F(s) = K_p + \frac{K_i}{s} + K_d s,$$

  where $K_p, K_i, K_d \geq 0$

- Lead-lag controller:

$$F_y(s) = F_r(s) = F(s) = K\left(\frac{\tau_D s + 1}{\beta\tau_D s + 1}\right)\left(\frac{\tau_I s + 1}{\tau_I s + \gamma}\right),$$

  where $K, \tau_D, \tau_I > 0$ and $0 \leq \beta, \gamma < 1$

- State-feedback controller with observer:

$$\begin{aligned} F_r(s) &= \left(1 - L(sI - A + KC + BL)^{-1}B\right)\ell_0 \\ F_y(s) &= L(sI - A + KC + BL)^{-1}K \end{aligned}$$

## Discrete-time state-space forms

A continuous time system with zero-order-hold input signal and sample period $T$ can be written in discrete-time as:

$$\boxed{\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ y(k) &= Hx(k) \end{aligned}}$$

where

$$\begin{aligned} F &= e^{AT} \\ G &= \int_{\tau=0}^{T} e^{A\tau} d\tau B = [\text{if } A^{-1} \text{ exists}] = A^{-1}(e^{AT} - I)B \\ H &= C \end{aligned}$$

# Problem 1: basic questions (6/30)

Answer only *'true'* or *'false'*. Each correct answer gives 1 point, each wrong answer gives $-1$ point. Minimum total points for Part A and B is 0, respectively.

## Part A

*Note:* Write 'skip' if your total home assignment score $\geq 8$

i) The system

$$G(s) = \frac{s+2}{s^2+s-2}$$

is input-output stable.

ii) The following system is non-minimum phase

$$G(s) = \frac{e^{-2s}}{s+2}.$$

iii) A system on the following state space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

has a transform function

$$G(s) = \frac{s}{s^2+2s+1}$$

**(3 p)**

## Part B

*Note:* Write 'skip' if your total home assignment score $\geq 12$

i) The transfer function of a closed-loop system, with state feedback control, differs when we add an observer.

ii) Given a controllable system, the poles of an observer can be placed arbitrarily.

iii) The bandwidth of the closed-loop system $G_c(s)$ determines how quickly it can respond to changes in the reference signal $r(t)$.

**(3 p)**

# Proposed solution to problem 1

## Part A

i) False. Compute the poles.

ii) True. Contains a time-lag.

iii) False. State-space description is on controllable canonical form and does not yield the same $G(s)$.

## Part B

i) False. $G_c(s)$ is the same in both cases.

ii) False. A controllable system can place the poles of state-feedback controller arbitrarily.

iii) True. $Y(i\omega) = G_c(i\omega)R(i\omega)$

## Problem 2 (6/30)

**a)** We wish to control the temperature $y(t)$ of a car heater using a voltage $u(t)$. We use a basic system model

$$G(s) = \frac{0.5}{s + 0.5}.$$

Use a P-controller and determine the closed-loop system $G_c(s)$. Choose $K$ such that the closed-loop system is stable.

**(3 p)**

**b)** Suppose we want to maintain a temperature given by the step

$$r(t) = \begin{cases} r_0, & t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Determine the corresponding response $y(t)$ of the closed-loop system and sketch how the temperature evolves when $K = 1$ versus $K = 3$.

**(3 p)**

# Proposed solution to problem 2

**a)** Deriving the closed-loop system we have

$$G_c = \frac{GF}{1+GF} = \frac{0.5K}{s+0.5+0.5K}$$

with a single pole at $s_1 = -0.5(1+K)$. For stability we require that $\mathrm{Re}\{s_1\} < 0$, or equivalently $K > -1$.

**b)** The step response of the system is given by

$$y(t) = \mathcal{L}^{-1}\{G_c(s)R(s)\} = \mathcal{L}^{-1}\left\{\frac{0.5K}{s+0.5+0.5K}\frac{r_0}{s}\right\}$$

For $K = 1$ we get

$$y(t) = 0.5r_0\mathcal{L}^{-1}\left\{\frac{1}{s+1}\frac{1}{s}\right\} = 0.5r_0(1-e^{-t})$$

using the Laplace tables. Similarly, for $K = 3$, we get

$$y(t) = 1.5r_0\mathcal{L}^{-1}\left\{\frac{1}{s+2}\frac{1}{s}\right\} = 0.75r_0(1-e^{-2t})$$

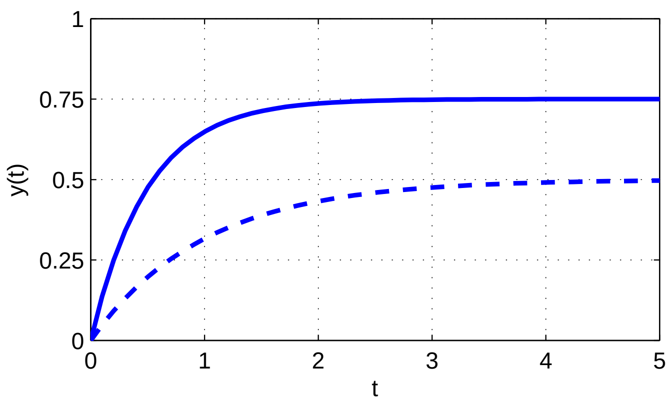



Figure 1: Response $y(t)$ when $r_0 = 1$. $K = 1$ is dashed, $K = 3$ is solid.

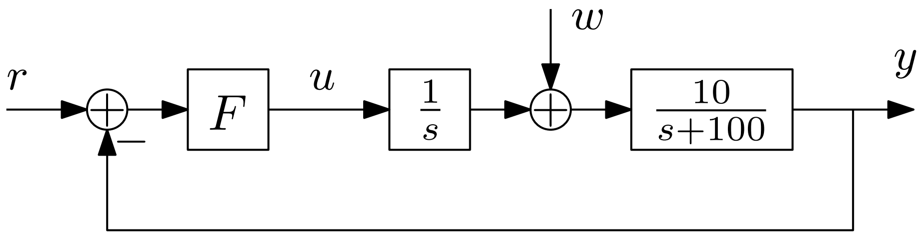



Figure 2: System with internal disturbance $w$ and feedback controller $F(s)$

# Problem 3 (6/30)

**a)** Consider a system with an internal disturbance denoted $w(t)$ as shown in Figure 2. We use feedback control with $F(s) = K$ that ensures the closed-loop stable is system. Suppose the reference is $r(t) \equiv 0$ and that the disturbance is a step

$$w(t) = \begin{cases} w_0, & \text{for } t \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

What is the final value of the output $y(t)$ in this setting?

**(4 p)**

**b)** Suppose that we are controlling a mechanical system based on a model $G(s)$ in a stable manner. The complementary sensitivity function is

$$T(s) = \frac{8s}{s + 100}.$$

As the actual system $G_0(s)$ wears down, it may differ substantially from our assumed model. By measuring the frequency response of the system $G_0(i\omega)$ we estimate that it differs from $G(i\omega)$ by at most 10% over all $\omega$. Is the closed-loop system still stable?

**(2 p)**

# Proposed solution to problem 3

**a)** Let

$$G_1 = \frac{1}{s} \quad \text{and} \quad G_2 = \frac{10}{s+100}$$

Then we can write

$$Y = G_2\left(W + FG_1(R - Y)\right)$$

Solving for $Y$ when $R \equiv 0$ and $F \equiv K$ gives

$$Y = \frac{10s}{s^2 + 100s + 10K} W.$$

Assuming that the closed-loop system is stable, we can use the final value theorem to get

$$y_f = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \frac{10s}{s^2 + 100s + 10K} \frac{w_0}{s} = 0.$$

Thus the effect of the disturbance vanishes.

**b)** The experiments tell us that the relative model errors are bounded

$$|\Delta_G(i\omega)| \leq 0.10.$$

Assuming that the designed controller yielded stability for the model $G(s)$ and that $G(s)$ shares the same number of unstable poles as the unknown system $G_0(s)$, we know that if

$$|\Delta_G(i\omega)| < \frac{1}{|T(i\omega)|}$$

holds, then the closed-loop system is still stable.

We have

$$\frac{1}{|T(i\omega)|} = \frac{\sqrt{\omega^2 + 100^2}}{8|\omega|}$$

and using the extremes $\omega : 0 \to \infty$ we see that $1/8 < \frac{1}{|T(i\omega)|}$. Thus relative model errors will not exceed the limit, and the closed-loop system is still stable.

# Problem 4 (6/30)

**(a)** Pair each step response in Figure 3 (A-D) with the correct pole zero plot in Figure 4 (I-IV).

**(1.5 p)**

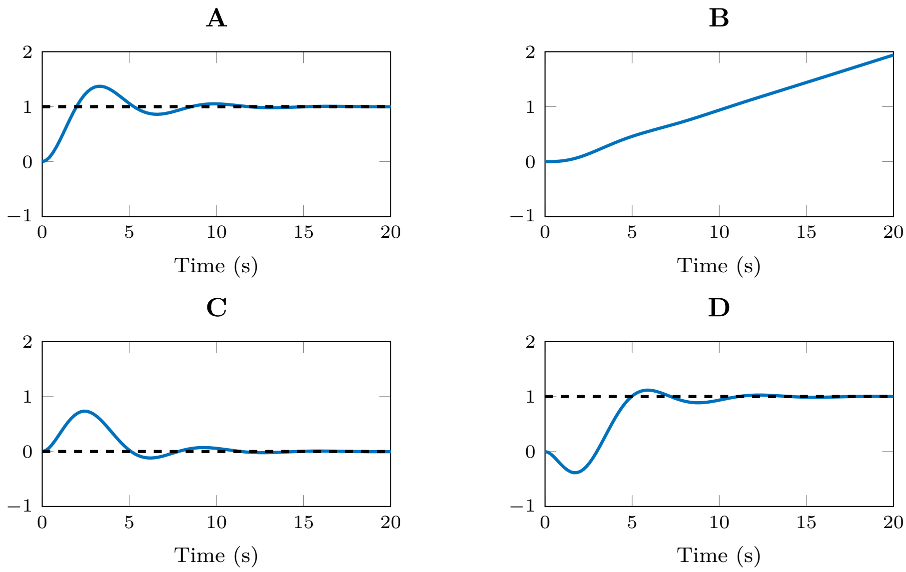



Figure 3: Step responses of four different systems in problem 4 (a) (dashed line represents the final value of $y(t)$)

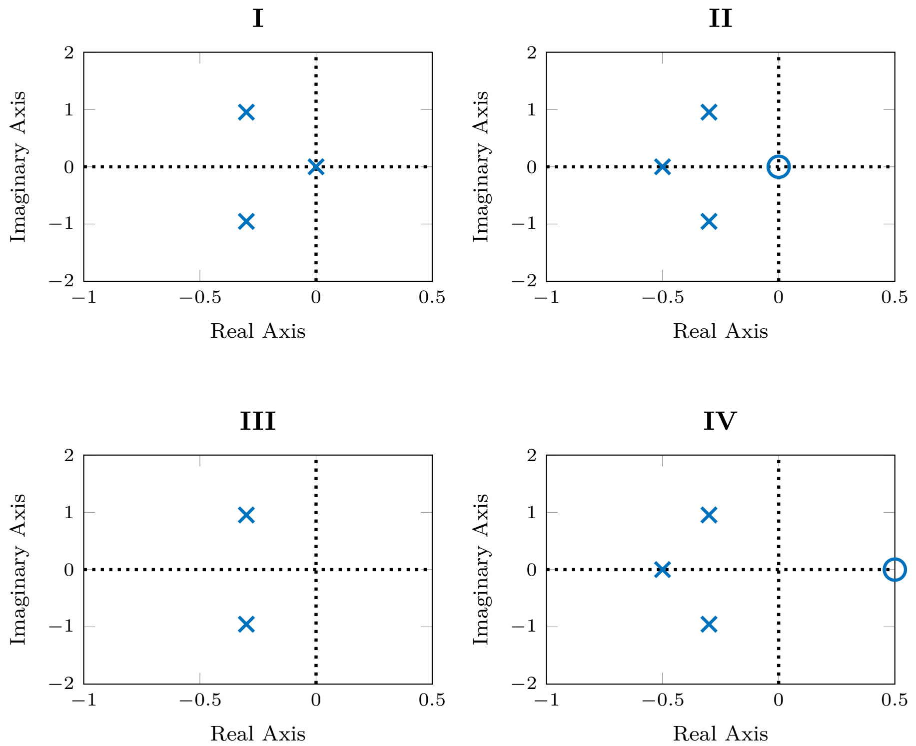



Figure 4: Pole zero maps for problem 4 (a)

**(b)** A former student of Introduction to Computer Controlled Systems is working at ABB studying a small part of an industrial robot. The Bode diagram of the system $G(s)$ is shown in Figure 5.

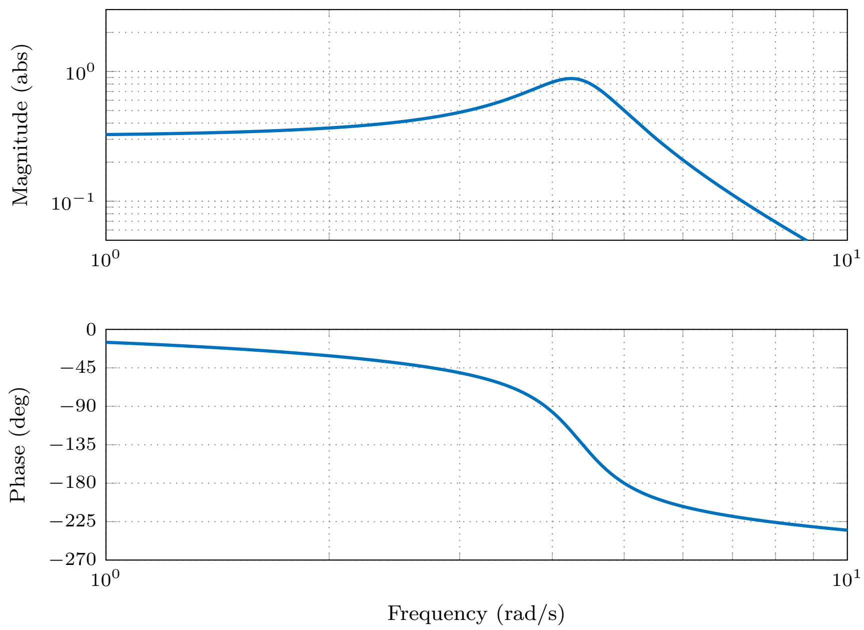



Figure 5: Bode diagram of the industrial process $G(s)$ in problem 4 (b).

To control the system $G(s)$, she uses a proportional controller, $F(s) = K$, and slowly increases $K$ starting from zero.

(i) Is the closed-loop system $G_c(s)$ stable for small values of $K$?

**(0.5 p)**

(ii) At a certain $K = K_0$, the output of the closed-loop system $G_c(s)$ oscillates with constant frequency and amplitude despite the reference signal $r(t)$ being constant. What is the value of $K_0$ and what is the frequency of the oscillation?

**(1 p)**

(iii) After the initial trail, it is concluded that the proportional controller cannot fulfill the requirements on the system. Design a PD controller

$$F(s) = K(1 + T_d s),$$

i.e. give $K$ and $T_d$, such that the open-loop system $G_o(s) = F(s)G(s)$ gets crossover frequency[1] $w_c$ that is the same as the frequency in (ii) and phase margin[2] $\varphi_m = 30°$. (If you have not solved (ii), you can use $\omega_c = 6$.)

**(3 p)**

[1] sv: skärfrekvens
[2] sv: fasmarginal

# Proposed solution to problem 4

(a) The step response in B is not stable can be paired with I that has a pole at the origin. Step response C tends to zero as time tend to infinity and must have a zero at the origin, i.e. II. D goes in the "wrong" direction and therfore has a zero in the right half plane like IV. That leaves A and III.

**Answer:**

| | | |
|---|---|---|
| A | — | III |
| B | — | I |
| C | — | II |
| D | — | IV |

(b) (i) Yes, the closed loop system is stable for small $K$ since $|G_o(i\omega)| = K|G(i\omega)| < 1$ for all frequencies. Hence, the Nyquist curve cannot encircle $-1$.

(ii) The phase curve crosses $-180°$ at $\omega_c = 5$ rad/s. Since $G_o(i\omega) = KG(i\omega)$, $K$ does not affect the phase curve but only shifts the amplitude curve. We get $|G(i\omega_c)| = |G(i5)| = 0.5$ and setting $K = 2$ will shift the magnitude curve such that $|G_o(i5)| = 1.$ and the Nyquist curve of the loop gain will pass through $-1$ and the system becomes marginally stable.

**Answer:** $K_0 = 2$ and $\omega_c = 5$ rad/s.

(iii) For the loop gain $G_o(i\omega)$, the phase is given by

$$\arg G_o(i\omega) = \arg F(i\omega)G(i\omega) = \arctan(T_d\omega) + \arg G(i\omega).$$

and the magnitude is given by

$$|G_o(i\omega)| = |F(i\omega)G(i\omega)| = K\sqrt{1 + T_d^2\omega^2}|G(i\omega)|.$$

From the text, we require $|G_o(i\omega_c)| = 1$ and $\arg G_o(i\omega_c) = -180+30 = -150°$. That is

$$\begin{cases} \arctan(T_d\omega_c) = -150° - \arg G(i\omega_c), \\ K = \dfrac{1}{\sqrt{1 + T_d^2 w_c^2}|G(i\omega_c)|}. \end{cases} \tag{1}$$

**Using $\omega_c = 5$:** The Bode diagram gives $|G(i5)| = 0.5$ and $\arg G(i5) = -180$. Inserting this in (1), we get $T_d = \frac{1}{5\sqrt{3}} = 0.12$ and $K = \sqrt{3} = 1.73$.

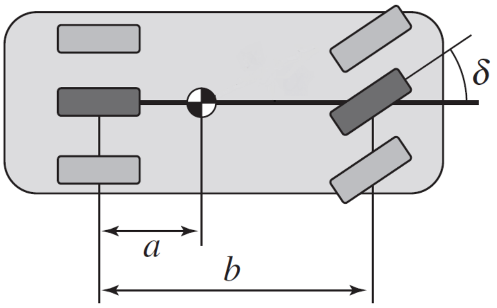
**Using $\omega_c = 6$:** The Bode diagram gives $|G(i6)| = 0.2$ and $\arg G(i6) = -210$. Inserting this in (1), we get $T_d = \frac{\sqrt{3}}{6} = 0.29$ and $K = 2.5$.

**Answer:**

$$\begin{cases} \omega_c = 5: \quad K = 1.73, \; T_d = 0.12 \\ \omega_c = 6: \quad K = 2.50, \; T_d = 0.29 \end{cases}$$

# Problem 5 (6/30)

A vehicle is controlled in 2 dimensions according to the diagram below:




The wheel base is $b$, the steering angle is $\delta$ and the distance of the rear wheels to the center of mass is $a$. After computing its dynamics and linearizing, the following system is obtained:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \gamma \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $x_1$ and $x_2$ are the horizontal and vertical 2D coordinates and $\gamma = \frac{a}{b}$. The input would be the steering angle $\delta$, and the output would be the horizontal position.

**a)** For $\gamma = 0.2$, we want to control the car so that its system poles are at -1. Compute the feedback matrix $L$ in continuous time

**(2 p)**

**b)** Discretize the state space model given above for a given sampling time $T$.

**(1 p)**

**c)** Assume $T = 0.1$. We can now only measure the output, so we need an observer in order to be able to estimate the states of the system. Compute the observer gain $K$ for the **discrete-time system** you computed in b). Choose the poles of the observer so that it follows the system dynamics properly.

**(3 p)**

# Proposed solution to problem 5

**(a)** To do so, we use the usual state feedback matrix:

$$\dot{x} = (A - BL)x$$

$A - BL$ has the characteristic polynomial: $\lambda^2 + 0.2\ell_1\lambda + \ell_2\lambda + \ell_1 = 0$ which should be equal to the one with the desired poles, that is $(\lambda + 1)^2$. Making the different powers equal, we get $\ell_1 = 1$ and $\ell_2 = 1.8$.

**(b)** We use now the usual ZOH sampling, which yields the following matrices:

$$F = e^{AT} \quad G = \int_0^T e^{A\tau} B d\tau$$

with $F$ being:

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

The inverse of $(sI - A)$ is given by:

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

Taking the inverse Laplace transform is straightforward, obtaining that:

$$F = e^{AT} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}$$

Once this is known, computing $G$ is straightforward taking into account that the integral operator can go inside each component of the resulting matrix:

$$G = \int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma \\ 1 \end{bmatrix} = \int_0^T \begin{bmatrix} \gamma + \tau \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma T + \frac{T^2}{2} \\ T \end{bmatrix}$$

and $H = C$.

**(c)** We do the same as a), but for the matrix $(F - KH)$. The characteristic polynomial is in this case $\lambda^2 + \lambda(k_1 - 2) + 1 - k_1 + 0.1k_2 = 0$. The poles taken above, translated to discrete-time, are $e^{-0.1}$. For example, taking the poles at -2 in continuous-time, i.e. $e^{-0.2}$ in discrete-time, we get the polynomial $\lambda^2 - 2e^{-0.2}\lambda + (e^{-0.2})^2$.
Making powers equal, we get $k_1 = 0.3625$ and $k_2 = 0.3286$.