Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: August 26, 2017

Venue: Bergsbrunnagatan 15, sal 2

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

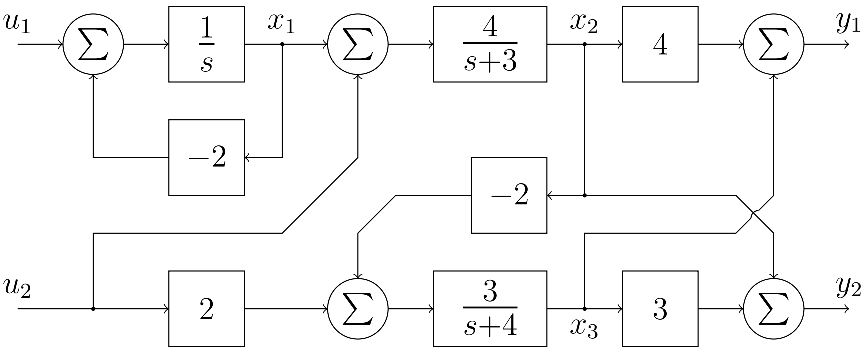
Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the spring semester 2017. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 The block diagram belows shows a system with two inputs and two outputs.



(a) Give the state space model of the system in the block diagram above, with $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$, $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ and $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ as input, output and state vector, and with x_1, x_2 and x_3 as in the block diagram. **(4p)**

(b) Is the state space model in (a) stable and/or a minimal realization? **(3p)**

Problem 2 Consider the discrete-time stochastic process

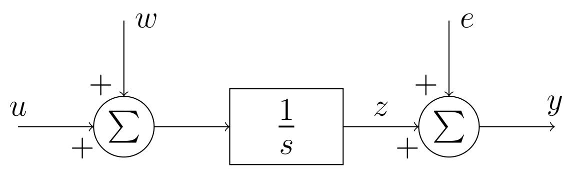
$$\begin{cases} x(k+1) = \begin{bmatrix} 0.4 & 1 \\ -0.6 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0.6 \\ 0 \end{bmatrix} v(k), \\ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v(k), \end{cases} \qquad Ev(k) = 0, \ \Phi_v(\omega) = 1. \quad (1)$$

(a) Determine the covariance matrix of the state vector, $\Pi_x = Ex(k)x(k)^T$, and the variance of the output, $Ey(k)^2$. **(3p)**

(b) What is the spectrum, $\Phi_y(\omega)$, of the output? **(3p)**

(c) Give the Kalman filter gain, K, for the $\hat{x}(k|k-1)$ estimator for (1). *Hint:* If you find it necessary you may use that $P = E\tilde{x}\tilde{x}^T$ is diagonal in this case. **(5p)**

Problem 3 A simple model describing variations of the level in a water reservoir is given in the block diagram below.



Here the input u is the controlled net flow into the reservoir, z is the level and y is the measured level. There are two independent, zero mean noise sources: the measurement noise, e, and uncontrolled variations in the net flow, w.

It is desirable to keep the variations of the level as small as possible, at a minimal cost in terms of control power. Therefore the LQG control law $u = -L\hat{x}$, which minimizes the cost function

$$V = E\left[z^2 + \rho^2 u^2\right], \qquad \rho > 0,$$

is used. (The estimate $\hat{x}$ is obtained from a Kalman filter, which is not considered in this problem.)

(a) In a first approximation it is assumed that both w and e are white noise. Find the optimal feedback gain L, expressed in ρ. (Use $x = z$ as state variable.) **(3p)**

(b) It turns out that

$$w(t) = \frac{1}{p+1} v(t),$$

where v is white noise, is a much better description of w. The measurement noise, e, is still assumed to be white. Give a state space representation of the total model, where the dynamics of w are incorporated. Use the "standard" form, i.e. determine the matrices and vectors A, B, N, M and C in

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t), \\ z(t) = Mx(t), \\ y(t) = Cx(t) + v_2(t). \end{cases}$$

Also, relate v_1 and v_2 to v and e. **(4p)**

(c) Find the optimal state feedback gain L for the total model in (b). **(4p)**

Problem 4 A double tank system (similar to the one in the MPC demo lab) has the linearized model

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t). \end{cases}$$

Here $y(t)$ is the water level in the lower tank and $u(t)$ is the flow into the upper tank (from a pump).

The controller is implemented as a sampling state feedback controller, with a zero-order hold (ZOH) circuit. That is,

$$u(t) = u(kh) \text{ for } kh \leq t < kh + h, \text{ and } u(kh) = -Lx(kh) + mr(kh), \quad (2)$$

where h is the sampling interval. Discrete-time control design then requires a discrete-time model of the system:

$$\begin{cases} x(kh+h) = Fx(kh) + Gu(kh), \\ y(kh) = Hx(kh) \end{cases} \Leftrightarrow y(kh) = H(qI - F)^{-1}Gu(kh).$$

(a) Determine the ZOH sampled model of the system, i.e. give F, G and H in the state space model above. The answer should be expressed in the sampling period h. **(4p)**

(b) The state feedback gain L in (2) should be determined by use of the pole placement technique. It is found that the performance of the closed loop system would be satisfactory for the *continuous-time* pole polynomial $s^2 + 2s + 1$ (if a continuous-time controller is used).

What *discrete-time* closed loop pole polynomial, q^2+aq+b, should one aim for in order to have an equivalent performance when the sampling controller (2) is used? The answer should be expressed in the sampling period h. (You need not compute L.) **(2p)**

(c) For a certain choice of sampling period the ZOH sampled model is

$$y(kh) = \frac{0.5(q + 0.5)}{q^2 - 0.75q + 0.125} u(kh).$$

The desired closed loop pole polynomial is then $q^2 - 0.5q + 0.0625$. Determine the value of m in (2) such that there is unit static gain from r to y. **(1p)**

Problem 5 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) $\Phi(\omega) = \frac{4}{\omega^2 - 4}$ is the spectrum of a continuous-time stochastic process.

(b) The discrete-time system $y(k) = \frac{0.2}{q-0.8}u(k)$ has unit static gain.

(c) White noise is characterized by that its covariance function $r(\tau)$ is constant and non-zero for all time lags τ.

(d) For a Kalman filter, based on a correct model and full knowledge of the noise, the *output innovations* are white noise.

(e) For a linear time-invariant (LTI) system LQG design yields an LTI controller.

(f) With MPC it is not possible to handle bounds on the control input.

(g) In MPC the computational load typically increases with the *control/input horizon.*

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 6 *The HW bonus points (from the spring 2017) are exchangeable for this problem.*
Consider the continuous-time stochastic process

$$\dot{x}(t) = Ax(t) + Nv(t), \quad A = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}, \quad N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Ev(t) = 0, \quad \Phi_v(\omega) = 32.$$

(a) What is the covariance matrix, $\Pi_x^c = Ex(t)x(t)^T$, of the state vector for this stochastic process? **(2p)**

(b) A discrete-time model representation of the stochastic process above is

$$x(kh+h) = Fx(kh) + w(kh), \quad F = e^{Ah}, \quad R_w = Ew(kh)w(kh)^T = \begin{bmatrix} r_1 & r_{12} \\ r_{12} & r_2 \end{bmatrix}.$$

Compute the matrix F when the sampling period is $h = 0.5$. **(1p)**

(c) In the discrete-time model the process noise is white and vector valued, and its covariance R_w is a 2×2 matrix. The discrete-time model is such that the covariance matrix for its state vector is identical to the one for the continuous-time stochastic process, i.e., $\Pi_x^d = Ex(kh)x(kh)^T = Ex(t)x(t)^T = \Pi_x^c$. Compute R_w so that this property holds. **(4p)**

Solutions to the exam in Automatic Control II, 2017-08-26:

1. (a) From the block diagram we get

$$\begin{aligned}
x_1 &= \frac{1}{p}(-2x_1 + u_1) &&\Leftrightarrow \quad px_1 = -2x_1 + u_1,\\
x_2 &= \frac{4}{p+3}(x_1 + u_2) &&\Leftrightarrow \quad px_2 = 4x_1 - 3x_2 + 4u_2,\\
x_3 &= \frac{3}{p+4}(-2x_2 + 2u_2) &&\Leftrightarrow \quad px_3 = -6x_2 - 4x_3 + 6u_2.
\end{aligned}$$

We also see that $y_1 = 4x_2 + x_3$ and $y_2 = x_2 + 3x_3$. All together we get

$$\begin{cases}
\dot{x}(t) = \overbrace{\begin{bmatrix} -2 & 0 & 0 \\ 4 & -3 & 0 \\ 0 & -6 & -4 \end{bmatrix}}^{=A} x(t) + \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 6 \end{bmatrix}}^{=B} u(t),\\
y(t) = \underbrace{\begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}}_{=C} x(t).
\end{cases}$$

(b) The poles are the eigenvalues of A. Since A is triangular all eigenvalues are on the diagonal, and these are all negative. Hence the system is stable. Minimal realization $\Leftrightarrow$ both controllable and observable:

$$\mathcal{S} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & \\ 0 & 4 & 4 & \cdots \\ 0 & 6 & 0 & \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 3 \\ 16 & -18 & -12 \\ & \vdots & \end{bmatrix}$$

Both $\mathcal{S}$ and $\mathcal{O}$ have full rank (it suffices to check the three first columns and rows). Thus, the system is both controllable and observable, and thereby a minimal realization.

2. (a) The covariance matrix solves the discrete-time Lyapunov equation, $\Pi_x = F\Pi_x F^T + NRN^T$ (can be used directly since v is white noise), where $R = 1$ in this case. Here we have

$$F = \begin{bmatrix} 0.4 & 1 \\ -0.6 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}. \quad \text{Setting} \quad \Pi_x = \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix}$$

and spelling out the Lyapunov equation, the following equation system is obtained:

$$\begin{cases}
\pi_1 = 0.16\pi_1 + 0.8\pi_{12} + \pi_2 + 0.36,\\
\pi_{12} = -0.24\pi_1 - 0.6\pi_{12},\\
\pi_2 = 0.36\pi_1,
\end{cases}
\Leftrightarrow
\begin{cases}
\pi_1 = \dfrac{0.36}{0.6} = 0.6,\\
\pi_{12} = -\dfrac{0.24}{1.6} \cdot 0.6 = -0.09,\\
\pi_2 = 0.36 \cdot 0.6 = 0.216.
\end{cases}$$

Since $y = Hx + v$ we get $Ey^2 = E[(Hx+v)(Hx+v)^T] = H\Pi_x H^T + R_v = \pi_1 + R_v$ (since $x(k)$ and $v(k)$ are uncorrelated). Hence

$$\Pi_x = \begin{bmatrix} 0.6 & -0.09 \\ -0.09 & 0.216 \end{bmatrix}, \quad Ey^2 = \pi_1 + R_v = 0.6 + 1 = 1.6.$$

(b) The output spectrum is $\Phi_y(\omega) = |G(e^{i\omega})|^2 \Phi_v(\omega)$. The system (1) is on observer canonical form so we get $G(q) = \frac{0.6q}{q^2-0.4q+0.6} + 1 = \frac{q^2+0.2q+0.6}{q^2-0.4q+0.6}$ (can also use $G(q) = H(qI-F)^{-1}N + 1$), and since $\Phi_v(\omega) = 1$ we have

$$\begin{aligned}
\Phi_y(\omega) &= \frac{e^{-i2\omega} + 0.2e^{-i\omega} + 0.6}{e^{-i2\omega} - 0.4e^{-i\omega} + 0.6} \cdot \frac{e^{i2\omega} + 0.2e^{i\omega} + 0.6}{e^{i2\omega} - 0.4e^{i\omega} + 0.6} \\
&= \frac{1 + 0.2^2 + 0.6^2 + (0.2 + 0.2\cdot 0.6)(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{i2\omega})}{1 + (-0.4)^2 + 0.6^2 + (-0.4 - 0.4\cdot 0.6)(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{i2\omega})} \\
&= \frac{1.40 + 0.32(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{-i2\omega})}{1.52 - 0.64(e^{i\omega} + e^{-i\omega}) + 0.6(e^{i2\omega} + e^{-i2\omega})} = \frac{1.40 + 0.64\cos\omega + 1.2\cos 2\omega}{1.52 - 1.28\cos\omega + 1.2\cos 2\omega}.
\end{aligned}$$

(c) Notice that (1) is on *innovations form* if $F - NH$ is stable (has all eigenvalues inside the unit circle). Here $\det(qI - F + NH) = q^2 + 0.2q + 0.6$ $\Rightarrow$ eigenvalues in $-0.1 \pm i\sqrt{0.59}$, which is inside the unit circle. Hence the system is on innovations form, and then the Kalman gain is $K = N$ — see Eq. (5.83) in Glad/Ljung (this holds for discrete-time systems as well). Thus, without solving any DARE we can directly state that

$$K = N = \begin{bmatrix} 0.6 \\ 0 \end{bmatrix}.$$

(Of course it is possible and correct to obtain K by solving the DARE. The solution then is $P = 0$! Note though that $R_1 = R_2 = R_{12} = 1$!)

3. (a) A state space representation is

$$\dot{x} = u + w, \quad z = x, \quad y = x + e.$$

Theorem 9.1 $\Rightarrow$ $L = Q_2^{-1}B^TS$, where $S = S^T > 0$ solves the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here $Q_1 = 1$, $Q_2 = \rho^2$, $A = 0$, $B = 1$ and $M = 1$, leading to

$$0 = 1 - S^2/\rho^2 \qquad \Rightarrow \qquad S = \rho,$$

and then $L = \rho^{-2}\rho = 1/\rho$.

(b) We have that $pw = -w + v$, and therefore

$$\begin{cases} \dot{z} = w + u, \\ \dot{w} = -w + v, \\ z = z, \\ y = z + e \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v, \\ z = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + e, \end{cases}$$

with $x = \begin{bmatrix} z & w \end{bmatrix}^T$.

(c) The CARE becomes

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} + \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rho^{-2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{bmatrix},$$

which, taken element by element, leads to the equation system

$$\begin{cases} 0 = 1 - s_1^2/\rho^2, \\ 0 = s_1 - s_{12} - s_1 s_{12}/\rho^2, \\ 0 = 2s_{12} - 2s_2 - s_{12}^2/\rho^2, \end{cases} \quad \Rightarrow \quad \begin{cases} s_1 = \rho, \\ s_{12} = \dfrac{s_1}{1 + s_1/\rho^2} = \dfrac{\rho^2}{\rho + 1}, \\ s_2 = \left(2 - s_{12}/\rho^2\right) s_{12}/2 = \dfrac{\rho^2(2\rho + 1)}{2(\rho + 1)^2}. \end{cases}$$

The state feedback gain is then $L = \rho^{-2} \begin{bmatrix} s_1 & s_{12} \end{bmatrix} = \begin{bmatrix} 1/\rho & 1/(\rho + 1) \end{bmatrix}$.

4. (a) Use Theorem 4.1. Then we need e^{At}, which is obtained by the Laplace transform:

$$\mathcal{L}[e^{At}] = (sI - A)^{-1} = \begin{bmatrix} s+1 & 0 \\ -1 & s+0.5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)(s+0.5)} & \frac{1}{s+0.5} \end{bmatrix},$$

$$\Leftrightarrow \quad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2(e^{-0.5t} - e^{-t}) & e^{-0.5t} \end{bmatrix}$$

Hence,

$$F = e^{Ah} = \begin{bmatrix} e^{-h} & 0 \\ 2(e^{-0.5h} - e^{-h}) & e^{-0.5h} \end{bmatrix}, \quad H = C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

$$G = \int_0^h e^{At} B dt = \int_0^h \begin{bmatrix} e^{-t} \\ 2(e^{-0.5t} - e^{-t}) \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-h} \\ 2 - 4e^{-0.5h} + 2e^{-h} \end{bmatrix}.$$

(b) For ZOH sampling continuous-time poles p are mapped on discrete-time poles e^{ph}. The continuous-time pole polynomial here corresponds to a double pole in -1, so the corresponding discrete-time poles should be a double pole in e^{-h}. This corresponds to the discrete-time pole polynomial $(q - e^{-h})^2 = q^2 - 2e^{-h}q + e^{-2h}$.

(c) The closed loop system will be

$$y(kh) = \frac{0.5(q + 0.5)}{q^2 - 0.5q + 0.0625} mr(kh) = G_c(q) mr(kh).$$

Static gain corresponds to setting $q = 1$. In order to have unit static gain we must have

$$1 = G_c(1)m = \frac{0.5(1 + 0.5)m}{1 - 0.5 + 0.0625} \quad \Leftrightarrow \quad m = \frac{1 - 0.5 + 0.0625}{0.5(1 + 0.5)} = \frac{0.5625}{0.75} = 0.75.$$

5. **(a)** False (A spectrum is always non-negative); **(b)** True (Set $q = 1$); **(c)** False (For white noise $r(\tau) = 0$ for $\tau \neq 0$); **(d)** True (Theorem 5.3?); **(e)** True (Theorem 9.1); **(f)** False (This is handled by the numerical optimization); **(g)** True (Control horizon = dimension of the optimization problem)

6. **(a)** Π_x^c solves the continuous-time Lyapunov equation, $0 = A\Pi_x^c + \Pi_x^c A^T + NR_vN^T$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} + \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 32 \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \Leftrightarrow$$

$$\begin{cases} 0 = -4\pi_1 + 32, \\ 0 = \pi_1 - 4\pi_{12}, \\ 0 = 2(\pi_{12} - 2\pi_2), \end{cases} \Leftrightarrow \begin{cases} \pi_1 = 8, \\ \pi_{12} = 2, \\ \pi_2 = 1, \end{cases} \Leftrightarrow \quad \Pi_x^c = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}.$$

(b) Use that $\mathcal{L}[e^{At}] = (sI - A)^{-1}$:

$$(sI - A)^{-1} = \begin{bmatrix} s+2 & 0 \\ -1 & s+2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+2)^2} & \frac{1}{s+2} \end{bmatrix} \quad \Leftrightarrow \quad e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ te^{-2t} & e^{-2t} \end{bmatrix}$$

Thus

$$F = \begin{bmatrix} e^{-1} & 0 \\ 0.5e^{-1} & e^{-1} \end{bmatrix} = e^{-1} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix}.$$

(c) Use the discrete-time Lyapunov equation, $\Pi_x^d = F\Pi_x^d F^T + R_w$ and that $\Pi_x^d = \Pi_x^c \Rightarrow$

$$R_w = \Pi_x^c - F\Pi_x^c F^T = \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix} - e^{-1} \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} e^{-1} = \begin{bmatrix} 8(1 - e^{-2}) & 2 - 6e^{-2} \\ 2 - 6e^{-2} & 1 - 5e^{-2} \end{bmatrix}.$$