Exam in Automatic Control II
Reglerteknik II 5hp (1RT495)

Date: December 19, 2017

Venue: Bergsbrunnagatan 15, sal 1

Responsible teacher: Hans Rosth.

Aiding material: Calculator, mathematical handbooks, textbooks by Glad & Ljung (Reglerteori/Control theory & Reglerteknik). Additional notes in the textbooks are allowed.

Preliminary grades: 23p for grade 3, 33p for grade 4, 43p for grade 5.

Use separate sheets for each problem, i.e. no more than one problem per sheet. Write your exam code on every sheet.

Important: Your solutions should be well motivated unless else is stated in the problem formulation! Vague or lacking motivations may lead to a reduced number of points.

Problem 6 is an alternative to the homework assignments from the autumn semester 2017. (In case you choose to hand in a solution to Problem 6 you will be accounted for the best performance of the homework assignments and Problem 6.)

Good luck!

Problem 1 The velocity of a moving object is modelled as

$$Y(s) = \frac{1}{s+1} U(s). \tag{1}$$

The system is to be controlled by a zero-order-hold (ZOH) sampling controller. That is, the input is piece-wise constant between the sampling instants, so

$$u(t) = u(kh) \qquad \text{for} \qquad kh \le t < kh + h,$$

where h is the sampling interval.

(a) In order to analyze the closed loop system a dicrete-time model of the system is needed. Show that the ZOH sampled model of (1) is

$$y(kh) = \frac{\beta}{q - \alpha} u(kh),$$

and give the values of α and β (expressed in h). **(2p)**

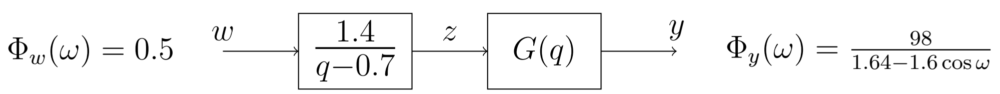
(b) The proportional feedback

$$u(kh) = 3(r(kh) - y(kh))$$

is used, where r is the reference signal. Determine for which sampling intervals $h > 0$ the closed loop system is stable. **(4p)**

(c) Assume that the proportional control in (b) is used, and that the sampling interval is chosen so that the closed loop system is stable. Determine the final value of the output, $\lim_{k \to \infty} y(kh)$, when $r(kh)$ is a unit step. **(2p)**

Problem 2 The block diagram below represents a stationary discrete-time stochastic process. The transfer operator $G(q)$ is minimum phase (with $G(1) \ge 0$), and w is zero mean white noise.



(a) Determine the spectrum for z. **(2p)**

(b) Determine the transfer operator $G(q)$. **(4p)**

Problem 3 Specify for each of the following statements whether it is true or false. No motivations required — only answers "true"/"false" are considered!

(a) The discrete-time system $y(k) = \frac{0.8}{q+0.8}u(k)$ has unit static gain.

(b) The sampling frequency is 50% of the Nyquist frequency.

(c) In MPC the *control/input horizon* is normally much longer than the *prediction/output horizon.*

(d) In MPC the computational load increases with the *control/input horizon.*

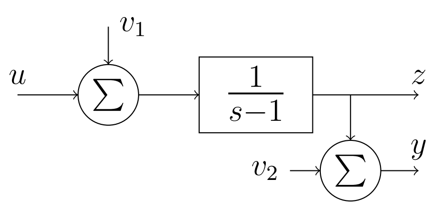
(e) An MPC controller can only be implemented as a continuous-time controller.

(f) A commonly used model of white noise has the spectrum $\Phi(\omega) = \frac{\omega^2}{\omega^4+4}$.

(g) $\Phi(\omega) = \frac{4}{1-2\cos\omega}$ is the spectrum of a discrete-time stochastic process.

Each correct answer scores +1, each incorrect answer scores −1, and omitted answers score 0 points. (Minimal total score is 0 points.) **(7p)**

Problem 4 The block diagram below shows an unstable continuous-time system.



The system is affected by the process noise v_1 and the measurement noise v_2. The noise processes, v_1 and v_2, are zero mean, uncorrelated white noise processes with intensities $R_1 = 3$ and $R_2 = 1$.

The system is to be stabilized by the standard LQG controller

$$u(t) = -L\hat{x}(t), \tag{2}$$

that minimizes the cost function

$$V = E\left[8z^2 + u^2\right]. \tag{3}$$

(a) Give the "standard form" state space representation for the system. **(1p)**

(b) Determine the Kalman filter that provides the estimate $\hat{x}(t)$. **(3p)**

(c) Determine the feedback gain L in (2). **(3p)**

(d) What are the poles of the closed loop system? **(2p)**

(e) Determine the value of the cost function V in (3). **(3p)**

Problem 5 A certain industrial process is described by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k),$$

where $v(k)$ is zero mean white noise, and $Ev(k)^2 = 1$. Initially both state variables were measured, but due to a malfunctioning sensor the only available measurement now is

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + e(k),$$

where $e(k)$ is zero mean white noise, uncorrelated to $v(k)$ and with intensity $Ee(k)^2 = r$.

(a) Since the value of $x_1(k)$ is required for subsequent calculations needed in the production line, the missing measurements are replaced by the estimate $\hat{x}_1(k) = Ex_1(k) = 0$ (for all k). What is the variance of the estimation error $\tilde{x}_1(k) = x_1(k) - \hat{x}_1(k)$ for this estimate? **(3p)**

(b) Is $x_1(k)$ observable from $y(k)$? **(1p)**

(c) A clever student from Uppsala suggests that a Kalman filter should be used instead for estimation of $x_1(k)$[1]. Show that

$$P = \begin{bmatrix} p_1 & 1 \\ 1 & 1 \end{bmatrix}$$

is the solution of the associated algebraic Riccati equation, and determine p_1. **(4p)**

(d) Determine the variance of the estimation error $\tilde{x}_1(k)$ when the Kalman filter in (c) is used. Is the estimate improved? **(2p)**

Problem 6 *The HW bonus points (from the autumn 2017) are exchangeable for this problem.*
A system with one input and two outputs is described by

$$\begin{cases} \dot{y}_1(t) + 2y_2(t) = 2\dot{u}(t) + u(t), \\ \dot{y}_2(t) + y_2(t) + 2y_1(t) = u(t). \end{cases}$$

(a) Give a state space representation for the system. **(4p)**

(b) Is your state space representation in (a) a minimal realization? **(2p)**

(c) Is the system stable? **(1p)**

[1] The Kalman filter will of course provide an estimate of the full state vector, but then x_1 is part of that.

Solutions to the exam in Automatic Control II, 2017-12-19:

1. **(a)** Set up a state space representation and use Theorem 4.1:

$$\begin{cases} \dot{x} = -x + u, \\ y = x, \end{cases} \quad \Rightarrow \quad \begin{cases} x(kh+h) = e^{-h}x(kh) + (1-e^{-h})u(kh), \\ \quad y(kh) = x(kh). \end{cases}$$

Thus,

$$y(kh) = H(qI-F)^{-1}Gu(kh) = \frac{1-e^{-h}}{q-e^{-h}}u(kh) \quad \Rightarrow \quad \alpha = e^{-h}, \quad \beta = 1-e^{-h}.$$

(b) The closed loop system is

$$y(kh) = \frac{G_o(q)}{1+G_o(q)}r(kh) = \frac{3\frac{\beta}{q-\alpha}}{1+3\frac{\beta}{q-\alpha}}r(kh) = \frac{3\beta}{q-\alpha+3\beta}r(kh),$$

whose pole is $\alpha - 3\beta = 4e^{-h} - 3$. For stability the pole must be inside the unit circle:

$$-1 < 4e^{-h} - 3 < 1 \quad \Leftrightarrow \quad 2 < 4e^{-h} < 4 \quad \Leftrightarrow \quad 0.5 < e^{-h} < 1.$$

Since $e^{-h} < 1$ for all $h > 0$ it is $0.5 < e^{-h}$ that will restrict h:

$$0.5 < e^{-h} \quad \Leftrightarrow \quad \log 0.5 < -h \quad \Leftrightarrow \quad h < \log 2 \approx 0.693.$$

(c) Use the final value theorem:

$$\lim_{k\to\infty} y(kh) = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)G_c(z)R(z) = G_c(1),$$

since $R(z) = \frac{z}{z-1}$ (a unit step). From (a) we have that

$$G_c(z) = \frac{3\beta}{z-\alpha+3\beta} \quad \Rightarrow \quad G_c(1) = \frac{3\beta}{1-\alpha+3\beta} = \frac{3-3e^{-h}}{4-4e^{-h}} = \frac{3}{4}.$$

2. **(a)** We have

$$\Phi_z(\omega) = \left|\frac{1.4}{e^{i\omega}-0.7}\right|^2 \Phi_w(\omega) = \frac{1.4}{e^{i\omega}-0.7} \cdot \frac{1.4}{e^{-i\omega}-0.7} \cdot 0.5$$
$$= \frac{1.4^2 \cdot 0.5}{1+0.7^2-0.7(e^{i\omega}+e^{-i\omega})} = \frac{0.98}{1.49-1.4\cos\omega}.$$

(b) Again we can use that $\Phi_y(\omega) = |G(e^{i\omega})|^2\Phi_z(\omega)$, and from (a) we have $\Phi_z(\omega)$. Thus,

$$|G(e^{i\omega})|^2 = \frac{\Phi_y(\omega)}{\Phi_z(\omega)} = \frac{\frac{98}{1.64-1.6\cos\omega}}{\frac{0.98}{1.49-1.4\cos\omega}} = \frac{100(1.49-1.4\cos\omega)}{1.64-1.6\cos\omega}$$
$$= 100 \cdot \frac{1.4}{1.6} \cdot \frac{-\frac{1.49}{1.4}+\cos\omega}{-\frac{1.64}{1.6}+\cos\omega}.$$

We try with

$$G(q) = K \cdot \frac{q+b}{q+a} \quad \Rightarrow \quad |G(e^{i\omega})|^2 = K^2 \cdot \frac{(e^{i\omega}+b)(e^{-i\omega}+b)}{(e^{i\omega}+a)(e^{-i\omega}+a)}$$

$$= K^2 \cdot \frac{1+b^2+b(e^{i\omega}+e^{-i\omega})}{1+a^2+a(e^{i\omega}+e^{-i\omega})} = K^2 \cdot \frac{1+b^2+2b\cos\omega}{1+a^2+2a\cos\omega} = K^2 \cdot \frac{b}{a} \cdot \frac{\frac{1+b^2}{2b}+\cos\omega}{\frac{1+a^2}{2a}+\cos\omega}.$$

Comparison with the expression above gives

$$\begin{cases} K^2 \cdot \dfrac{b}{a} = 100 \cdot \dfrac{1.4}{1.6}, \\ \dfrac{1+b^2}{2b} = -\dfrac{1.49}{1.4}, \\ \dfrac{1+a^2}{2a} = -\dfrac{1.64}{1.6}, \end{cases} \quad \Rightarrow \quad \begin{cases} b = \dfrac{-1.49 \pm 0.51}{1.4} = -0.7, \\ a = \dfrac{-1.64 \pm 0.36}{1.6} = -0.8, \\ K = (\pm)10. \end{cases}$$

Stationarity $\Leftrightarrow$ stability $\Leftrightarrow |a| < 1$, minimum phase $\Leftrightarrow |b| < 1$, and $G(1) \geq 0$ $\Leftrightarrow K \geq 0$. Thus $G(q) = 10 \cdot \frac{q-0.7}{q-0.8}$

3. (a) False (Here $G(1) = 0.8/1.8 \neq 1$); **(b)** False (The Nyquist frequency is per definition 50% of the sampling frequency); **(c)** False (The other way around); **(d)** True (The control horizon decides the dimension of the optimization problem); **(e)** False (The opposite); **(f)** False (White noise has flat/constant spectrum); **(g)** False (A spectrum is always non-negative — here $\Phi(0) = -4!$)

4. (a) From the block diagram we have

$$z = \frac{1}{p-1}(u+v_1) \quad \Leftrightarrow \quad pz = 2z + u + v_1, \quad \text{and} \quad y = z + v_2.$$

A natural choice of state variable is $x = z$, which gives

$$\begin{cases} \dot{x} = x + u + v_1, \\ z = x, \\ y = x + v_2. \end{cases}$$

(b) The Kalman filter is $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$, with $K = PC^TR_2^{-1}$ where $P = P^T \geq 0$ solves the CARE $0 = AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP$ (here $R_{12} = 0$ since v_1 and v_2 are uncorrelated). Here we have $A = 1$, $B = N = M = C = 1$ so the CARE spells out as

$$0 = 2P + 3 - P^2 \quad \Leftrightarrow \quad P^2 - 2P - 3 = 0 \quad \Leftrightarrow \quad P = 3.$$

Hence, $K = P = 3$.

(c) We have $L = Q_2^{-1}B^TS$, where $S = S^T \geq 0$ solves the CARE $0 = A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS$. Here the CARE spells out as

$$0 = 2S + 8 - S^2 \quad \Leftrightarrow \quad S^2 - 2S - 8 = 0 \quad \Leftrightarrow \quad S = 4,$$

and thus $L = S = 4$.
(d) The poles of the closed loop system are the the zeros of the pole polynomial

$$0 = \det(sI - A + BL)\det(sI - A + KC) = (s-1+4)(s-1+3) = (s+3)(s+2),$$

so the poles are -3 and (the observer pole) -2.
(e) Compute $V = E[8z^2 + u^2] = 8Ez^2 + Eu^2$. A state space model of the closed loop system is

$$\begin{cases} \dot{x} = Ax + Bu + Nv_1 = Ax - BL\hat{x} + Nv_1, \\ \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) = KCx + (A - BL - KC)\hat{x} + Kv_2, \end{cases}$$

or, in block matrix form,

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x \\ \hat{x} \end{bmatrix}}_{=x_c} = \underbrace{\begin{bmatrix} A & -BL \\ KC & A - BL - KC \end{bmatrix}}_{=A_c}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \underbrace{\begin{bmatrix} N & 0 \\ 0 & K \end{bmatrix}}_{=N_c}\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{=v_c}.$$

Then the covariance matrix $\Pi_c = Ex_cx_c^T$ solves the continuous-time Lyapunov equation $0 = A_c\Pi_c + \Pi_cA_c^T + N_cR_cN_c^T$, where $R_c = Ev_cv_c^T$. As $u = -L\hat{x} = -\begin{bmatrix} 0 & L \end{bmatrix} x_c = -L_cx_c$ we get $Euu^T = EL_cx_cx_c^TL_c^T = L_c\Pi_cL_c^T$. Furthermore, as $z = Mx = \begin{bmatrix} M & 0 \end{bmatrix} x_c = M_cx_c$ we also get $Ezz^T = M_c\Pi_cM_c^T$.

We have $L = 4$ and $K = 3$, and thus

$$A_c = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix}, \quad N_c = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad \text{and} \quad R_c = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

The Lyapunov equation then is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix} + \begin{bmatrix} \pi_1 & \pi_{12} \\ \pi_{12} & \pi_2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2(\pi_1 - 4\pi_{12}) + 3 & 3\pi_1 - 5\pi_{12} - 4\pi_2 \\ 3\pi_1 - 5\pi_{12} - 4\pi_2 & 2(3\pi_{12} - 6\pi_2) + 9 \end{bmatrix},$$

which is solved by

$$\Pi_c = \begin{bmatrix} 4.5 & 1.5 \\ 1.5 & 1.5 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} Eu^2 = L_c\Pi_cL_c^T = 4^2 \cdot 1.5 = 24, \\ Ez^2 = M_c\Pi_cM_c^T = 4.5 \end{cases}$$

and thus $V = 8 \cdot 4.5 + 24 = 60$.
5. (a) Since $\hat{x}_1(k) = 0$, we have $\tilde{x}_1(k) = x_1(k)$, and thus

$$\tilde{x}_1(k+1) = 0.5\tilde{x}_1(k) + v(k).$$

The variance $E\tilde{x}_1(k)^2 = \Pi_{\tilde{x}}$ is then obtained from the discrete-time Lyapunov equation $\Pi_{\tilde{x}} = F\Pi_{\tilde{x}}F^T + GR_vG^T$. Here $F = 0.5$, $G = 1$ and $R_v = 1$, so

$$\Pi_{\tilde{x}} = 0.5^2\Pi_{\tilde{x}} + 1 \quad \Leftrightarrow \quad \Pi_{\tilde{x}} = \frac{1}{0.75} = \frac{4}{3}.$$

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathcal{O}\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = 0$$

regardless of x_1. Therefore it is *not* observable.

(c) The associated DARE is $P = FPF^T + NR_1N^T - FPH^T(HPH^T + R_2)^{-1}HPF^T$ (since $R_{12} = 0$), and here $R_1 = 1$ and $R_2 = r$, so written out the DARE becomes

$$\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$-\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + r\right)^{-1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix}\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25p_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{0.25p_{12}^2}{p_2+r} & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives the equation system

$$\begin{cases} p_1 = 0.25p_1 + 1 - \dfrac{0.25p_{12}^2}{p_2 + r}, \\ p_{12} = 1, \\ p_2 = 1, \end{cases} \quad \Rightarrow \quad P = \begin{bmatrix} p_1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$p_1 = 0.25p_1 + 1 - \frac{0.25}{1 + r} \quad \Leftrightarrow \quad p_1 = \frac{4}{3} - \frac{1}{3(1 + r)}.$$

(d) Since $P = E\tilde{x}\tilde{x}^T$, and $\tilde{x}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\tilde{x}$, we get

$$E\tilde{x}_1^2 = \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = p_1 = \frac{4}{3} - \frac{1}{3(1 + r)}.$$

Furthermore, $1 < p_1 < \frac{4}{3}$ for every $r > 0$, so the estimate from the Kalman filter *is* better than the estimate $\hat{x}_1 = 0$!

6. (a) There are several (infinitely many) possible state space representations. One possibility is the following choice of state variables: $x_1 = y_1 - 2u$ and $x_2 = y_2$. Then we can rewrite the differential equations as

$$\begin{cases} \dot{x}_1 + 2\dot{u} + 2x_2 = 2\dot{u} + u, \\ \dot{x}_2 + x_2 + 2x_1 + 4u = u \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}_1 = -2x_2 + u, \\ \dot{x}_2 = -2x_1 - x_2 - 3u, \\ y_1 = x_1 + 2u, \\ y_2 = x_2, \end{cases}$$

which in vector form, with $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$, is

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & -2 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u, \\ y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u. \end{cases}$$

Another possibility is to use the controller canonical form (which always works when there is only one input). Then we need to start by finding the transfer operator/function:

$$\begin{cases} py_1 + 2y_2 = (2p+1)u, \\ 2y_1 + (p+1)y_2 = u \end{cases} \Leftrightarrow \begin{bmatrix} p & 2 \\ 2 & p+1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2p+1 \\ 1 \end{bmatrix} u$$

$$\Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} p & 2 \\ 2 & p+1 \end{bmatrix}^{-1} \begin{bmatrix} 2p+1 \\ 1 \end{bmatrix} u = \frac{1}{p(p+1)-4}\begin{bmatrix} p+1 & -2 \\ -2 & p \end{bmatrix}\begin{bmatrix} 2p+1 \\ 1 \end{bmatrix} u$$

$$= \begin{bmatrix} \frac{2p^2+3p-1}{p^2+p-4} \\ \frac{-3p-2}{p^2+p-4} \end{bmatrix} u = \begin{bmatrix} 2 + \frac{p+7}{p^2+p-4} \\ \frac{-3p-2}{p^2+p-4} \end{bmatrix} u$$

The controller canonical form then gives us the state space representation

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 & 4 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y = \begin{bmatrix} 1 & 7 \\ -3 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u. \end{cases}$$

(b) A minimal realization is both controllable and observable. For the first state space representation in (a) we have

$$\mathcal{S} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ -3 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -2 \\ -2 & -1 \end{bmatrix}.$$

Obviously both $\mathcal{S}$ and $\mathcal{O}$ have full rank $\Leftrightarrow$ the state space representation is both controllable and observable, and hence a minimal realization.

The controller canonical form is obviously controllable. Thus we only need to check observability for that:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ -3 & -2 \\ 6 & 4 \\ 1 & -12 \end{bmatrix}.$$

Again $\mathcal{O}$ has full rank (eg. the first two rows are linearly independent), and we have observability, and thereby a minimal realization.

(c) The pole polynomial is $s^2 + s - 4$, and one pole is in the right half plane $\Rightarrow$ unstable.